Probability Stability and Monte Carlo Method

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***Abstract* – The method for the probability stability estimation of systems with randomly chosen parameters is presented in this paper. This method is simple, effective and can be applied in practice. The presented method provides the choice of those parameters for which the system has the maximum probability stability. In this paper Monte Carlo method was used to confirm the results obtained by the presented method for the probability stability estimation. The computational results are shown in tables.**

***Keywords* – Probability stability, Random parameter, Imperfect systems, Monte Carlo method.**

I. Introduction

Many systems with random parameters can be found in process industry, chemical industry, industry of plastic materials, the rubber industry, etc. Since the values of the stochastic parameters differ from wanted ones, the system can not work properly. These systems are known as imperfect systems and it is important to estimate the influence of parameters on the system performances in advance. This estimation is very important for the system stability, the quality of system work and the reliability of the system.

The well – known fact is that the stability of the system is determined by the value of the system parameters. If the parameters have constant values, the system is stable or nonstable depending on parameters values. If the parameters are stochastic, the system is stable with some probability called probability stability.

The basic methods for the probability stability estimation are given in [1-4]. These methods relate to the continuous systems. In [5] the method for the probability stability estimation of discrete systems with random parameters is presented. Some theorems from theory of random processes [6] and the basic condition for the discrete system stability, [7], are used. Also, the stability analysis for imperfect systems is given in [8].

In this paper the randomly chosen and time invariable parameters are considered only. This must be pointed up, because the parameters values can be randomly changed in time, also, under the influence of different factors, but this is not the matter of the research in this paper.

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The big importance of the presented method is in it application in practice. The selection of the adequate parameters values, for which the system has the largest probability stability, provides the stability of the system and correctness of system work. This method can be applied for different distributions of parameters such as uniform, normal, exponential, Poisson distribution and for the arbitrary order systems.

Monte Carlo method, [9], was used in this paper to confirm the results obtained by the presented method for the probability stability estimation. The results obtained by Monte Carlo method coincide with results obtained by the method for the probability stability estimation. The experiments were performed for the second and the third order systems with exponential and normal probability distribution of parameters. Monte Carlo method gives almost identical results like the method for the probability stability estimation. For the higher order systems and for the other probability distributions of parameters, Monte Carlo method provides very good results, also.

II. The Probability Stability Estimation of the Linear System

Let the discrete system is given by:

$$\sum_{i=0}^{n} l_i\, x(k+n-i) = u(k),\ \ l_0 = 1 \tag{1}$$

where l_i are random variables with probability distribution densities $p_i(l_i)$. It is necessary to determinate the probability stability of the roots of equation (1).

First, the stability region of the equation (1) in the parametric space is determinated. The characteristic polynomial of the equation (1) is:

$$z^n + l_1 z^{n-1} + \cdots + l_n = 0 \tag{2}$$

The necessary and sufficient condition for the stability of the difference equation roots is that all zeros of its characteristic polynomial are located inside the unit circle in the z – plane.

To test this condition the bilinear transformation method is used and the inside of the unit circle is mapped into the left half of the complex plane. Applying the Hurwitz criterion, the stability region, S_n, of difference equation (1) is obtained.

The system (1) is stable if all zeroes of the characteristic equation (2) are in the left half of the s –plane. The necessary and sufficient condition for the stability of system (1) is that all diagonal minors D_i of Hurwitz matrix D:

$$D = \begin{vmatrix} l_{n-1} & l_{n-3} & l_{n-5} & \cdots & 0 \\ l_n & l_{n-2} & l_{n-4} & \cdots & 0 \\ 0 & l_{n-1} & l_{n-3} & \cdots & 0 \\ 0 & l_n & l_{n-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & l_0 \end{vmatrix} \tag{3}$$

are greater than zero, i.e.:

$$D_1 = l_{n-1} > 0; \qquad D_2 = \begin{vmatrix} l_{n-1} & l_{n-3} \\ l_n & l_{n-2} \end{vmatrix} > 0, \text{ etc} \tag{4}$$

The stability region is obtained in the parametric space using the nonlinearities (4).

If parameters of the systems are independent variables, then the total density distribution is given by:

$$p(l_1,\ldots,l_n) = \prod_{i=1}^{n} p_i(l_i) \tag{5}$$

The probability stability of the system (1) is:

$$P = \underset{S_n}{\int \cdots \int} p(l_1,\ldots,l_n)\, dl_1 \cdots dl_n \tag{6}$$

where S_n is the stability region.

For the first order discrete system the stability region, S_1, is given by the:

$$-1 < l_1 < 1 \tag{7}$$

For the second order discrete system the stability region, S_2, in the parametric space l_1, l_2 is given by:

$$\begin{aligned} &1 - l_1 + l_2 \geq 0 \\ &1 + l_1 + l_2 \geq 0 \\ &l_2 \leq 1 \end{aligned} \tag{8}$$

The stability region S_2 is given on Fig.1 where N_2 presents the unstably region.

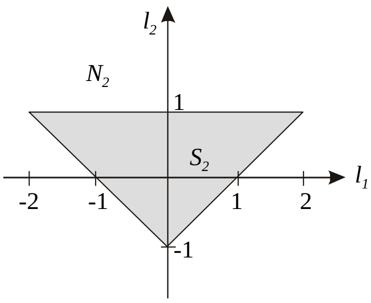

Fig.1. The stability region S_2 of the second order discrete system

In the case of the second order discrete system, the stability region is the triangle.

For the third order discrete system the stability region is obtained in the same way as at the second order discrete system and is given by the following relations:

$$\begin{aligned} &l_1 + l_2 + l_3 > -1 \\ &l_1 - l_2 + l_3 < 1 \\ &l_1 l_3 + 1 > l_2 + l_3^2 \end{aligned} \tag{9}$$

This stability region is given on Fig.2.

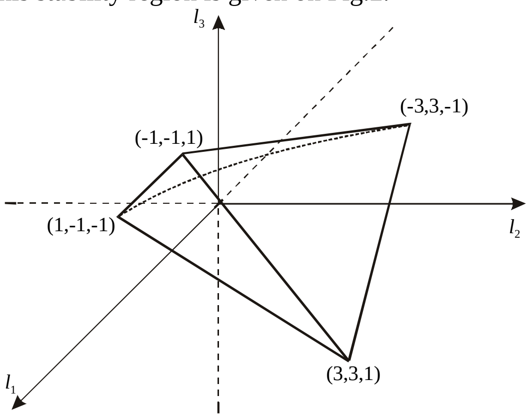

Fig.2. The stability region S_3 of the third order discrete system

For the n – th order system the region of stability is determinated using the relations (4), also. However, the calculation is too complex. The limits of the stability region are usually complex mathematical relations and is difficult to determinate the probability stability because it is necessary to integrate by the region of stability. Because of that is important to estimate the probability stability for the higher order systems for practical applications. For the probability stability estimation next theorems can be applied effectively.

Theorem 1. The stability region, S_n, of difference equation (1) belongs to the region $\overline{P}_n$ (hyper parallelepiped) given by:

$$|l_i| \leq \binom{n}{i} \tag{10}$$

The stability region is limited above by the region $\overline{P}_n$ in the parametric plane, $S_n \in \overline{P}_n$.

Theorem 2. The stability region, S_n, of difference equation (1) comprises the region $\underline{P}_n$ (simplex polyhedron) given by:

$$|l_1| + |l_2| + \cdots + |l_n| \leq 1 \tag{11}$$

The stability region is limited lower by the region $\underline{P}_n$, $\underline{P}_n \in S_n$.

The proofs of these theorems are given in [5]. According to the theorems, the probability stability can be estimated in the following way:

$$\iint\cdots\int_{\underline{P}_n} p(l_1,\ldots,l_n)dl_1\cdots dl_n < P < \iint\cdots\int_{\overline{P}_n} p(l_1,\ldots,l_n)dl_1\cdots dl_n$$

ie.:

$$P_{\underline{P}_n} < P < P_{\overline{P}_n} \tag{12}$$

where $P_{\underline{P}_n}$ is the probability that the stability region lies inside the region $\underline{P}_n$ and $P_{\overline{P}_n}$ is the probability that the stability region lies inside the region $\overline{P}n$.

Using these theorems, for the different types of parameters distributions, different formulas for the probability stability estimation are obtained, [5]. Using these formulas, the probability stability of the arbitrary order systems can be estimated. For the first, the second and the third order systems the probability stability can be calculated precisely, but for the higher order systems the estimation is performed using mentioned formulas.

III. THE PROBABILITY STABILITY CALCULATION USING MONTE CARLO METHOD

Monte Carlo method can be defined as statistical method, where statistical simulation is defined to be any method that utilizes sequences of random numbers to perform the simulation. Monte Carlo method gives approximate solution of different types of problems by performing statistical sampling experiments. This method can provide an approximate solution quickly and with the high level of accuracy, because the more simulation is performed, the more accurate approximation is obtained. Since this method gives only an approximate solution, the analysis of the approximation error is a major factor to take into account.

In this paper, Monte Carlo method is used to confirm the results obtained by the method for the probability stability estimation given in previous section. Monte Carlo method gives almost identical results like the method for the probability stability estimation.

The experiments were performed for the second and the third order systems with exponential and normal probability distribution of parameters. The random number generator was used to generate the values of the parameters with exponential and normal distribution. The experiment was performed on the 100.000 samples. The results are given in tables. Probabilities $P_{\underline{P}_n}$, P_{S_n} and $P_{\overline{P}_n}$ are obtained by the method for the probability stability estimation, and the probability of system stability, P, is calculated using the Monte Carlo method. The probabilities P_{S_n} and P have almost identical values and results correspond to the relation (12) which is the verification of correctness of proposed method for the probability stability estimation. The calculation of the probability stability using Monte Carlo method is much easier because there is no need to integrate by the region of stability, only the limits of stability region are required.-The probability stability is calculated as the quotient of the number of samples that belong to the region of stability and the number of all scanned samples. By samples we consider the values of system parameters. For the higher order systems and for the other probability distributions of parameters, Monte Carlo method provides very good results, also.

TABLE I
THE RESULTS OBTAINED FOR THE EXPONENTIAL DISTRIBUTION OF PARAMETERS FOR THE SECOND ORDER SYSTEM, $i=1,2$

l_i	0.1	0.5	0.8	1	1.5	2
$P_{\underline{P}_2}$	0.98657	0.3995	0.2159	0.1548	0.081	0.048
P	0.9999	0.7987	0.5806	0.4707	0.297	0.202
P_{S_2}	0.9999	0.7982	0.5820	0.4730	0.2994	0.201
$P_{\overline{P}_2}$	0.99995	0.8488	0.6549	0.5465	0.3604	0.248

TABLE II
THE RESULTS OBTAINED FOR THE EXPONENTIAL DISTRIBUTION OF PARAMETERS FOR THE THIRD ORDER SYSTEM, $i=1,2,3$

l_i	0.1	0.5	0.8	1	1.5	2
$P_{\underline{P}_3}$	0.23	0.1152	0.0395	0.0227	0.008	0.0036
P	0.9997	0.5932	0.3356	0.2355	0.1123	0.0606
P_{S_3}	0.9997	0.6115	0.3477	0.2462	0.1187	0.0644
$P_{\overline{P}_3}$	0.99995	0.8603	0.6803	0.5707	0.3662	0.2374

TABLE III
THE RESULTS OBTAINED FOR THE NORMAL DISTRIBUTION OF PARAMETERS FOR THE SECOND ORDER SYSTEM, $i=1,2$

l_i	0.2	0.3	0.4	0.4	0.6
σ_i	0.1	0.2	0.2	0.3	0.4
$P_{\underline{P}_2}$	0.9973	0.7078	0.4781	0.3959	0.1586
P	1	0.9994	0.9984	0.9686	0.8063
P_{S_2}	1	0.9995	0.9984	0.9680	0.8027
$P_{\overline{P}_2}$	1	0.9997	0.9986	0.9772	0.8411

TABLE IV
THE RESULTS OBTAINED FOR THE NORMAL DISTRIBUTION OF PARAMETERS FOR THE THIRD ORDER SYSTEM, $i=1,2,3$

l_i	0.2	0.3	0.4	0.4	0.6
σ_i	0.1	0.2	0.2	0.3	0.4
$P_{\underline{P}_3}$	0.7505	0.1807	0.0503	0.0663	0.0142
P	1	0.9739	0.9433	0.7919	0.4714
P_{S_3}	1	0.9735	0.9426	0.7937	0.4714
$P_{\overline{P}_3}$	1	0.9997	0.9986	0.9772	0.8413

IV. CONCLUSION

The method presented in this paper enables the probability stability estimation of systems with randomly chosen parameters. For systems with more random parameters and for systems for which the limits of the stability region in parametric space can be hardly approximated by linear functions, probability stability calculation is too complex and the probability stability estimation is required. Using this method is possibly to choose such values of parameters for which the system has the largest probability stability. The validity of the proposed method is approved by the well – known Monte Carlo method. The results obtained by both methods are almost identical and given in tables.

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