Accurate Algorithm for Calculating Characteristic Impedance of a Square Coaxial Line

Zaklina J. Mancic[1] and Vladimir V. Petrovic[2]

Abstract – **This paper presents an accurate algorithm (of maximal relative error $\delta_{\max} = 4 \cdot 10^{-6}$) for calculating the characteristic impedance of a transmission line of a square cross-section. The algorithm is based on known high accuracy approximations of elliptic integrals. The results for capacitance per unit length and characteristic impedance obtained in this way can be used as an accurate benchmark of a priori known accuracy.**

Keywords – ***Square coaxial line, Elliptic integrals.***

I. INTRODUCTION

For testing numerical methods and obtaining reference solutions of electromagnetic (EM) problems it is useful to have simple and accurate solutions of canonical problems (benchmark results). Moreover, it is important to know *a priori* the obtained accuracy [1]–[3]. According to the comparison of the results obtained by numerical methods and benchmark results an estimate can be made of the quality of the applied numerical methods. Such solutions are of the prime importance in all areas of the computational electromagnetics [2-11].

In our previous paper [10] we presented exact benchmark solution (based on exact values of elliptic integrals) and one approximate method for calculating per unit length capacitance (C') and characteristic impedance (Z_c) of the square coaxial line with homogeneous dielectric (Fig.1). This paper is the continuation of the research into more accurate and simple algorithms for this canonical, but also practical, problem.

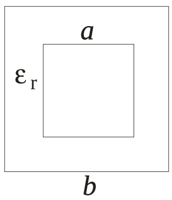

Fig.1. Coaxial line of the square cross section with homogeneous dielectric.

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II. ANALYTICAL SOLUTION VIA ELLIPTIC INTEGRALS

For the coaxial line from Fig.1, method of conformal mapping yields the exact analytical solution for normalized per unit length capacitance, $C'_{\mathrm{N}} = C'/(\varepsilon_0 \varepsilon_{\mathrm{r}})$, and for characteristic impedance Z_{c} that can be expressed (for nonmagnetic dielectric) as [4]

$$C'_{\mathrm{N}} = 8 f(k), \; Z_{\mathrm{c}} = \frac{\mu_0 c_0}{\varepsilon_{\mathrm{r}} C'_{\mathrm{N}}} = \frac{149896229}{10000000} \frac{\pi}{\sqrt{\varepsilon_{\mathrm{r}}} f(k)} [\Omega], \quad (1)$$

where

$$k = \left(\frac{\lambda' - \lambda}{\lambda' + \lambda}\right)^2, \; \lambda' = \sqrt{1-\lambda^2}, \; \lambda = f^{-1}(s), \; s = \frac{b+a}{b-a}, \quad (2)$$

$$f(k) = \frac{K(k)}{K'(k)}, \; K(k) = \int_0^{\pi/2} \frac{1}{\sqrt{1-k^2 \sin^2\theta}} d\theta . \quad (3)$$

Here, $K(k)$ is the complete elliptic integral of the first kind [12], [13], $K'(k) = K(k')$, $k' = \sqrt{1-k^2}$, variable k is called elliptical module, k' complementary module and f^{-1} denotes inverse function. Function $f(k)$, defined by (3), often appears in conformal mapping. It has an useful property,

$$f\left(\sqrt{1-k^2}\right) = f(k') = \frac{1}{f(k)}, \quad (4)$$

from which follows that if we known the value of f for some argument $k \in \left(0, 1/\sqrt{2}\right)$, $f(k) = s$, we automatically know the value of f for argument $k' = \sqrt{1-k^2}, k' \in \left(1/\sqrt{2}, 1\right)$, $f(k') = 1/s$. That means that it is enough to approximate function f only for $k \in \left(0, 1/\sqrt{2}\right)$. Then, for the rest of the interval, that is $k \in \left(1/\sqrt{2}, 1\right)$, formula (4) can be applied. We will denote the range $k \in \left(0, 1/\sqrt{2}\right)$ the basic range of the function f. The graph of the function $f(k)$ in this range in log-log scale is shown in Fig.2, and in the full range in Fig.3.

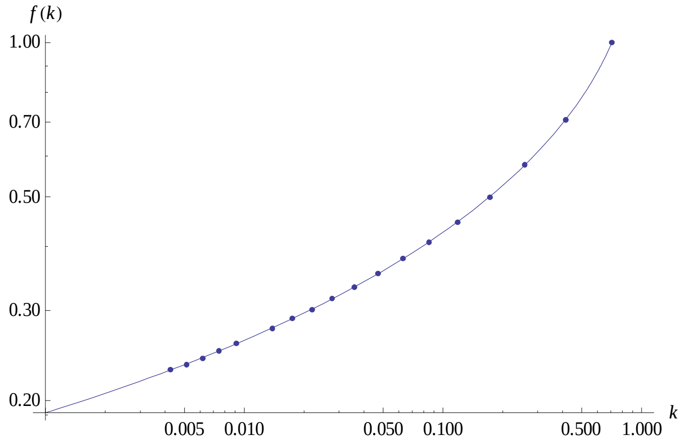

Fig.2. Graph of function $f(k)$ in log-log scale in its basic range, with several exact values marked by dots.

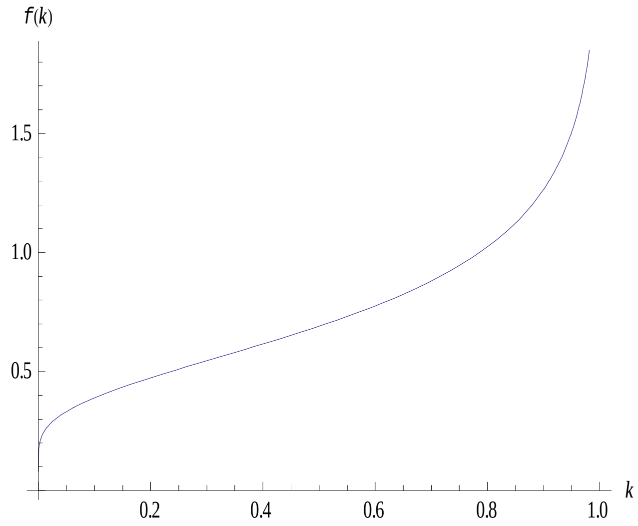

Fig.3. Graph of function $f(k)$ in its full range.

For a number of arguments k, expressed explicitly by the four basic operations, square root and cubic root, function $f(k)$ has explicit (exact) values [12]. Several such values (taken from [14]) are shown in Fig.2.

However, the main practical computational difficulty is calculation of the inverse function, $f^{-1}(s)$ for *arbitrary* argument s, as this function is usually not included in standard function packages (rare exceptions from this rule are [15] and [16]).

III. ACCURATE AND SIMPLE APPROXIMATE SOLUTION

To efficiently calculate C' and Z_c of the square coaxial line it is the best to have a simple and accurate analytical formulas for both $f(k)$ and $f^{-1}(s)$ and precise estimation of their accuracy. For this purpose we will apply a simple approximate formula from [1, eq.(17)], whose maximal relative error is $\delta_{\max} = 3\cdot 10^{-6}$,

$$f(k) \approx \begin{cases} \dfrac{\pi}{\ln\left(2\dfrac{1+\sqrt{k'}}{1-\sqrt{k'}}\right)}, & 0 \le k \le \dfrac{1}{\sqrt{2}},\ 0 \le f(k) \le 1 \\ 1/f(k'), & \dfrac{1}{\sqrt{2}} \le k \le 1,\ 1 \le f(k) \le \infty. \end{cases} \quad (5)$$

This formula has a very useful property, that can be explicitly solved for k or for k', thus enabling explicit expression for the inverse function too, as

$$f^{-1}(s) = \begin{cases} \sqrt{1-\left(\dfrac{e^{\pi/s}-2}{e^{\pi/s}+2}\right)^4}, & 0 \le s \le 1 \\ \sqrt{1-\left(f^{-1}(s^{-1})\right)^2}, & s > 1. \end{cases} \quad (6)$$

In this manner, we constructed a complete algorithm for calculating approximate C' and Z_c via elementary functions only, which consists of equations (1)–(3), (5) and (6). Next we proceed to analyze the relative error of this algorithm.

First, the relative error of the formula (5) throughout the full range of k is obtained by comparing the numerical results derived by it with the results of high accuracy calculation done by the Wolfram Mathematica [16]. From these results, relative error of the formula (6) is obtained and, by using the standard calculation of error propagation, the relative error of C' and Z_c for arbitrary ratio a/b is obtained. Results are shown in Fig.4. It can be seen that the maximal relative error of the algorithm is $\delta_{B,\max} = 4\cdot 10^{-6}$ and that the relative error away from this maximum quickly drops to very low values, thus showing the overall high accuracy of the algorithm.

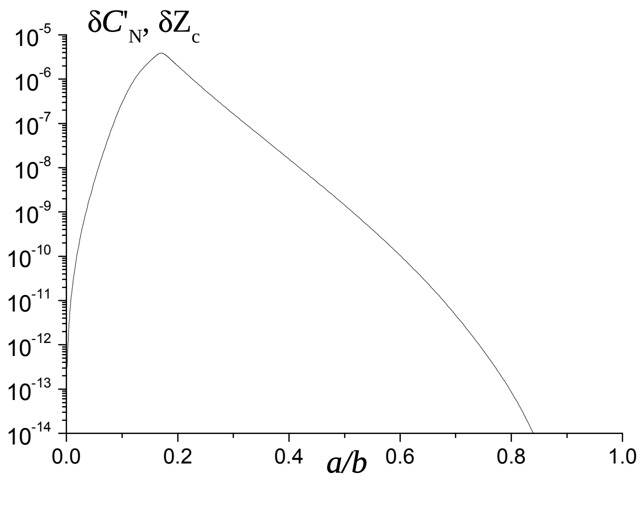

Fig.4. Relative error for normalized per unit length capacitance and for characteristic impedance.

As a further demonstration of the algorithm, values of C'_{N} and Z_{c0} (for the vacuum line, $\varepsilon_{\mathrm{r}} = 1$) obtain by it are shown in Figs.5 and 6.

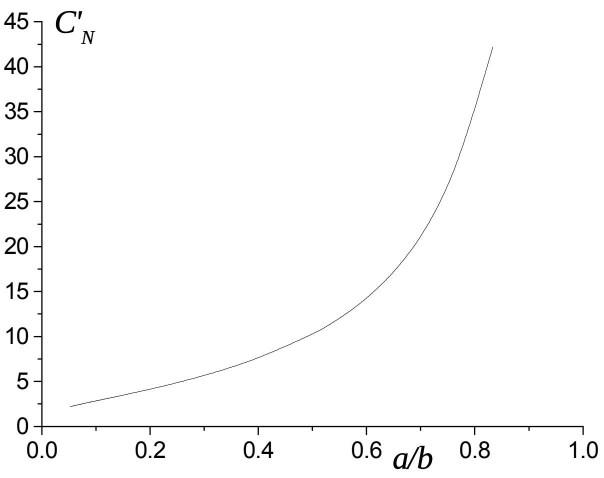

Fig.5. Dependence of C'_{N} on the ratio a/b.

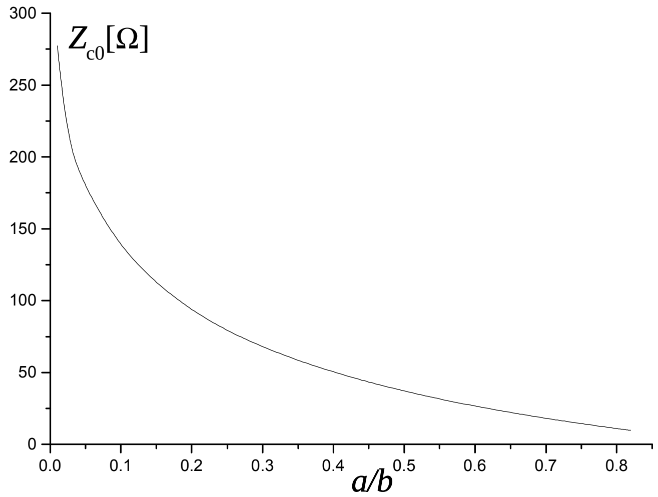

Fig.6. Dependence of Z_{c0} on the ratio a/b.

IV. Numerical results

In order to demonstrate the practical operation of the presented algorithm, Table I shows the intervals of absolute certainty of C'_{N} for a/b ranging from 0.01 to 0.80. In this table $C'_{\mathrm{N,estim}}$ denotes the value calculated by the presented algorithm. In addition, $C'_{\mathrm{N,min}}$ and $C'_{\mathrm{N,max}}$ denote the interval in which (with 100% certainty) the exact value of C'_{N} lies. It can be seen that the majority of the presented values for $C'_{\mathrm{N,estim}}$ have 7 to 8 accurate decimal digits, which is in agreement with the behavior of the relative error shown in Fig.4.

Table I. Intervals of absolute certainty for C'_{N}.

a/b	$C'_{\mathrm{N,min}}$	$C'_{\mathrm{N,estim}}$	$C'_{\mathrm{N,max}}$
0.01	1.391559	1.391559	1.391559
0.02	1.643946	1.643947	1.643946
0.03	1.839053	1.839053	1.839053
0.04	2.008149	2.008149	2.008149
0.05	2.162367	2.162368	2.162368
0.06	2.307132	2.307132	2.307132
0.07	2.445556	2.445556	2.445557
0.08	2.579625	2.579626	2.579626
0.09	2.710702	2.710703	2.710704
0.10	2.839775	2.839777	2.839778
0.15	3.476790	3.476799	3.476808
0.20	4.134479	4.134487	4.134495
0.25	4.844417	4.844420	4.844422
0.30	5.632827	5.632828	5.632829
0.35	6.527455	6.527457	6.527458
0.40	7.561524	7.561524	7.561524
0.45	8.777792	8.777792	8.777792
0.50	10.234093	10.234093	10.234093
0.55	12.012490	12.012490	12.012490
0.60	14.234880	14.234880	14.234880
0.65	17.092054	17.092054	17.092054
0.70	20.901582	20.901582	20.901582
0.75	26.234915	26.234915	26.234915
0.80	34.234907	34.234907	34.234907

V. Conclusion

The paper presents an accurate and simple algorithm for calculating per unit length capacitance and characteristic impedance of a square coaxial line. This problem can be regarded as one of the important canonical problems of the 2D electromagnetics. It can be applied for validating numerical methods and results of high and *a priori* known accuracy can serve as an excellent benchmark. Our algorithm is based on the known approximation of elliptic integrals that is computationally very simple and also enables derivation of both elliptic functions required for the computation. Maximal relative error of the results obtained by this algorithm is found to be $\delta_{\max} = 4\cdot 10^{-6}$.

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