Deriving measures for the goodness of 1-D spline interpolator filters

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Abstract – **The main topic of the work of the authors was deriving measures for the goodness of modified B-spline interpolators of degrees 5, 3 and 1. They have been optimized, through their degrees of freedom, in order to obtain better interpolator filters. The first measure is minimizing the energy in the stop band of the interpolator, with nonlinear constraints about the ripples in the frequency response. The second measure is minimizing the maximum value of the error kernel of the interpolator from 0 to some α_e.**

Keywords – **spline interpolartor filters, nonlinear constrains,**

I. INTRODUCTION

In The splines theory is well studied during the last years. The modified splines have some very attractive properties – maximum order with minimum support, very easy implementation in discrete structures, designed for processing of continuous data. The frequency responses of the interpolators, based on modified splines, have better attenuation in the stop band; compared to other interpolators, let's say Cubic Lagrange. The errorkernels of the modified spline interpolators are very close to the errorkernel of the sinc interpolator, which is the ideal one. The signal to noise ratio (SNR) and peak signal to noise ratio (PSNR) of the resized and resampled images with spline interpolators are very high, compared to the same case, involving other interpolators.

Some very good studies of splines are given in [1÷4].

II. SPLAIN THEORY

The normalized B-spline functions of degree *M* have *M+2* equally spaced nodes. For M odd the nodes are placed at the integers while for M even they are placed at the half-integers. The B-splines centered at the origin are defined as follows:

$$\beta^M(x) = \sum_{i=0}^{M+1} \frac{(-1)^i}{M!} \binom{M+1}{i} \left(x + \frac{M+1}{2} - i \right)_+^M, \tag{1}$$

where x_+^n is the truncated power function $\max(0,x)^n$. They belong to the class C^{M+1}, i.e. they are continues and have M-1 continues derivatives. The B-spline functions can be obtained recursively starting with the B-spline of zero degree as follows:

$$\beta^M = \beta^0 * \beta^{M-1}, \tag{2}$$

where:

$$\beta^0(x) = \begin{cases} 1 \; for \; |x| \le \frac{1}{2} \\ 0 \; for \; |x| > \frac{1}{2} \end{cases}. \tag{3}$$

The corresponding frequency-domain characteristics are given by:

$$B^M(2\pi f) = \left(\frac{\sin(\pi f)}{\pi f} \right)^{M+1}. \tag{4}$$

The continuous B-spline dilated by an integer scale factor $m \ge 1$ and of degree *M* is defined by:

$$\beta_m^M(x) = \frac{1}{m} \beta^M\left(\frac{x}{m}\right). \tag{5}$$

The dilated B-spline can be subsequently sampled:

$$b_m^M[k] = \frac{1}{m} \beta^M\left(\frac{k}{m}\right). \tag{6}$$

The discrete B-spline of degree M at the integer scale m is given by:

$$u_m^M = \overbrace{u_m^0 * u_m^0 ... u_m^0}^{M+1}, \tag{7}$$

where:

$$u_m^0 = \frac{1}{m} \overbrace{[1,1,...,1]}^{m}. \tag{8}$$

The relation between dilated, sampled and discrete B-splines of degree *M* is established through the following so-called *m*-scale relation:

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$$\beta_m^M(x) = \left(u_m^M * \beta_1^M\right)(x)$$
$$b_m^M[k] = (u_m^M * b_1^M)[k]. \qquad (9)$$

A form of the modified B-spline, which combines a central B-spline of degree n with its derivatives as given by

$$\varphi(x) = \sum_{k=0}^{N} \gamma_k \frac{d^k}{dx^k} \beta^N(x). \qquad (10)$$

The support of the modified B-splines (the approximation order as well) up to any number N is given by the following modified B-spline basis functions:

$$\beta^{\text{mod}}(x) = \sum_{n=0}^{N} \sum_{m \in Z} \gamma_{nm} \beta^n(x-m) \,. \qquad (11)$$

The function β^{mod} is the generating function for certain spline-like space V (β^{mod}). The scaling coefficients 'gammas' are actually the so called degrees of freedom, which provide the opportunity to adjust the modified B-spline approximation properties in an application-oriented manner.

In this report we are interested in the combination of 5-th, 3-rd, and
1-st degree B-splines. In this case, the modified B-spline basis function becomes:

$$\begin{aligned} &\beta^{\text{mod}}(t) = \beta^5(t) + \gamma_{30}\beta^3(t) + \\ &+ \gamma_{31}\beta^3(t-1) + \gamma_{31}\beta^3(t+1) + \\ &+ \gamma_{10}\beta^1(t) + \gamma_{11}\beta^1(t-1) + \gamma_{11}\beta^1(t+1) + \\ &+ \gamma_{12}\beta^1(t-2) + \gamma_{12}\beta^1(t+2). \end{aligned} \qquad (12)$$

This combination is of particular interest since the quintal term is very smooth with short support and by adding shifted and centered cubic and linear terms the model is improving the sharp signal parts.

Keeping the partition of unity condition gives $\gamma_{30} = -2\gamma_{31} - \gamma_{10} - 2\gamma_{11} - 2\gamma_{12}$. This implies that the latter equation looks like shown above:

$$\begin{aligned} &\beta^{\text{mod}}(t) = \beta^5(t) + \\ &+ \gamma_{31}\left[\beta^3(t-1) - 2\beta^3(t) + \beta^3(t+1)\right] + \\ &+ \gamma_{10}\left[\beta^1(t) - \beta^3(t)\right] + \\ &\gamma_{11}\left[\beta^1(t-1) + \beta^1(t+1) - 2\beta^3(t)\right] + \\ &+ \gamma_{12}\left[\beta^1(t-2) + \beta^1(t+2) - 2\beta^3(t)\right]. \end{aligned} \qquad (13)$$

That is nothing but a combination of a smoothing function and a difference part. By optimizing in am appropriate manner the parameters $\gamma_{31}, \gamma_{10}, \gamma_{11}, \gamma_{12}$ we can trade between smoothing and sharpening. Figure (1) shows combination between a centered B-spline of 5-rd degree, centered and shifted B-spline of 3-rd degree, and centered and shifted B-splines of 1-st degree.

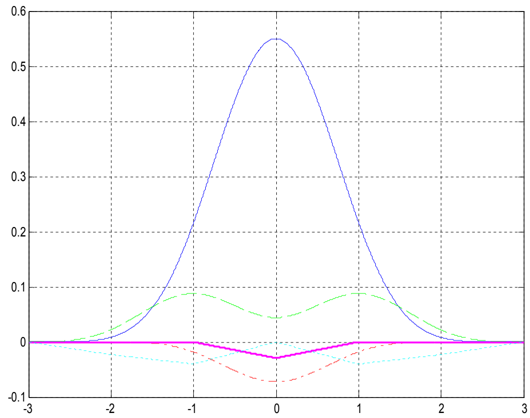

Fig.1. Combination of 5-th, 3-rd and 1-st degrees B-splines

III. Interpolation Theory

The main algorithm of interpolation due to splines is shown in the following equation:

$$s(x) = \sum_{k \in Z} c(k) \beta^n(x-k), \qquad (14)$$

Which involves the integer shifts of the central B-spline of degree n denoted by $\beta^n(x)$; the parameters of the model are the B-spline coefficients c (k), $s(x)$ are the values of the interpolated signal.

To derive this type of signal processing algorithm, we take the signal values $s(x)$, and we want to determine the coefficients c(k) of the B-spline model such that we have a perfect fit at the integers, i.e. $\forall k \in Z$:

$$\sum_{l \in Z} c(l) \beta^n(x-l)\Big|_{x=k} = s(k) \,. \qquad (15)$$

Using the discrete B-splines, this constraint can be rewritten in the form of a convolution:

$$s(k) = (b_1^n * c)(k). \qquad (16)$$

Defining the inverse convolution operator:

$$\left(b_1^n\right)^{-1}(k) \overset{n}{\longleftrightarrow} 1/B_1^n(z). \qquad (17)$$

The solution is found by inverse filtering

$$c(k) = \left(b_1^n\right)^{-1} * s(k). \qquad (18)$$

Since b_1^n is symmetric FIR (finite impulse response), the so-called direct B-spline filter $\left(b_1^n\right)^{-1}$ is an all-pole system that can be implemented very efficiently using a cascade of first-order causal and anti-causal recursive filters. This algorithm is stable numerically and is faster and easier to implement than any other numerical technique.

IV. ERROR KERNEL

There are two ways of estimating the properties of the optimized interpolators. The firs way is taking a set of pictures – Barbara, Lena, Hotel, Mandrill, Patterns, and part of the Geometrical image. Then each picture is successively rotated 10 times by 36 grades with the optimized interpolators. Finally, the SNR and the PSNR are computed for each rotated picture with respect to the original one. The latter operations are performed by the program "Spline Explorer", developed by Atanas Gotchev and Grigor Marchokov. The second way is by computing the so called errorkernels for each optimized modified combination of B-splines. Here is a brief explanation:

The main model of image acquisition consists in the following. Consider the one-dimensional function *r(x)* representing a true scene along the x axis. As a result of optical blurring, modeled as a linear convolution with a kernel *s(x)*, we obtain another continues function:

$$g_a(x) = (r * s)(x). \tag{19}$$

For the sake of simplicity we assume that the initial sampling grid is placed on the integer coordinates and generate the discrete sequence $g[k] = g_a(x)$. In continues time it can be modeled by the product $g_a comb(x)$, where:

$$comb(x) = \sum_{k=-\infty}^{\infty} \delta(x-k). \tag{20}$$

If we want to insert an additional parameter *u*, showing the relative position of the sampling device with respect to the function *s(x)*:

$$g_a(x-u) = (r * s)(x-u). \tag{21}$$

Sampling at integers and reconstructing by a continuous interpolation function gives:

$$y_a(x,u) = \sum_{k=-\infty}^{\infty} g_a(x-u) h(x-k). \tag{22}$$

Here, *h(x-k)* represents the impulse response of the interpolator, i.e. of the combination of modified B-splines. The reconstruction function also depends of the sampling phase and it is not simply a function of the difference *x-u*. The error between *g(x-u)* and *y(x, u)* called by the authors sampling and reconstructing (SR) blur, is a function of *u* as well:

$$\varepsilon_{SR}^2(u) = \int_{-\infty}^{\infty} (g_a(x-u) - y(x,u))^2 dx. \tag{23}$$

The average value of the SR blur plays the most important role in the estimation of the properties of the interpolator. It is equal to

$$E\{\varepsilon_{SR}^2\} = \int_{-\infty}^{\infty} \eta^2(2\pi f) \left|G_a(2\pi f)\right|^2 df, \tag{24}$$

where:

$$\eta^2(2\pi f) = \left|1 - H(2\pi f)\right|^2 + \sum_{k \neq 0} \left|H(2\pi(f-k))\right|^2. \tag{25}$$

Here, the function $H(2\pi f)$ represents the frequency response of the interpolator. The non-negative errorkernel η^2 quantifies the amount of errors introduced by the reconstruction kernel. The plotting variable α_e (see Fig.2) is defined by the equation $\alpha_e = \frac{f}{F_s}$, where F_s is the sampling frequency. It is not possible to achieve errorkernel equal to zero, if the sampling frequency is lower than 1 (or equivalently, if the cut-off frequency is higher than 0.5 for fixed unity sampling frequency). For low frequencies, i.e. for the pass band, the reconstruction function must be unity in order to avoid blurring.

A convenient expression of the errorkernel in the case of the modified B-splines can be written as:

$$\eta^2(j2\pi f) = \frac{\left(P(e^{j2\pi f})\right)^2 - 2P(e^{j2\pi f})\Phi(2\pi f) +}{\left(P(e^{j2\pi f})\right)^2} + \\ + \frac{\sum_{k=-\infty}^{\infty}\left(\Phi(2\pi(f-k))\right)^2}{\left(P(e^{j2\pi f})\right)^2}. \tag{26}$$

Where $\Phi(2\pi f)$ is the Fourier Transform of the continuous kernel, and $P(e^{j2\pi f})$ is the Fourier transform of the discrete kernel of the combination of B-splines.

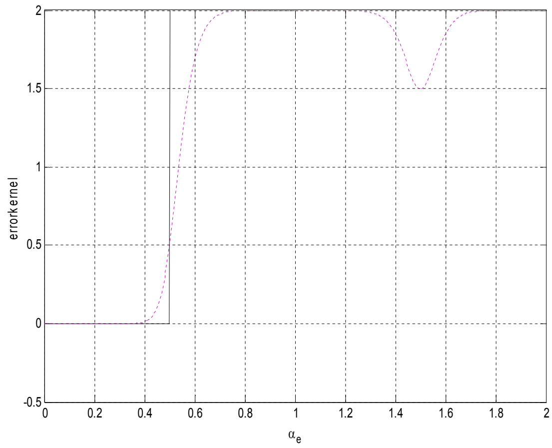

Fig.2. Errorkernels for sinc interpolator (solid line) and for quintal B-spline

The weighting coefficients in (13) are optimized according to two criteria – minimizing the energy in the stop band of the interpolators, and minimizing the errorkernel from *0* to some α_e .

V. Conclusion

Two of the most important unwanted effects, which occur in the interpolated image due to non-ideal interpolator, are aliasing and imaging. Aliasing is the effect of the appearing of unwanted frequencies as a result of the repetition of the original spectrum around multiplies of 2π . Imaging is an effect, which occurs for example by up sampling of the image by some factor *L*. After the up sampling in the pass band of the original spectrum appears together with *L-1* 'images'. The unwanted frequencies can interfere into the pass band during the process of resampling as a result of non-sufficient suppression of the frequency replicas during the previous step of continuous restoration.

It is the stop band of the interpolator, which is responsible for suppressing the frequencies, which cause the above mentioned effects.

In this report, the terms "modified B-splines" and "combination of B-splines" will be used as synonyms.

By the use of the term "interpolator" we mean an interpolator, based on the modified B-splines kernel.

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Acknowledgement

For the acknowledgement use the unnumbered section layout.

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