Dispersive TLM Z-transform based 3D Model of Left-Handed Metamaterials

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Abstract – **In this paper, numerical three-dimensional (3D) model of electromagnetic left-handed metamaterials (LH MTM), is derived by using the Transmission Line Matrix (TLM) method based on Z-transforms techniques. Model, implemented in time-domain TLM solver, is capable to account for frequency dispersive behaviour of LH MTM permittivity and permeability described by the Drude function. Model accuracy and efficiency are verified on some characteristic examples in comparison with analytical solutions.**

Keywords –**metamaterials**, **TLM method, Z-transforms**, **numerical dispersive model**

I. INTRODUCTION

Metamaterials (MTM) are broadly defined as an artificial media with extreme values of effective permittivity and permeability. These artificial electromagnetic (EM) structures exhibit highly unusual properties not readily found in nature, such as the antiparallelism between phase/group velocities and the reversal of classical phenomena (Doppler effect, Vavilov-Cherenkov radiation, Snell's law, Goos–Hanchen effect) [1,2]. In recent years, metamaterials, particularly the left-handed metamaterials (LH MTM) with negative refractive index (i.e. ε, $\mu < 0$), have been the subject of numerous researches that promise to bring about important technological and scientific advancements in telecommunications, radars and defense, nanolithography with light, microelectronics and medical imaging.

In order to validate specific EM properties and fundamental physics of LH MTM and to design advanced components based on metamaterials, a number of numerical techniques have been used. Some of them have been enhanced in the form of numerical model of metamaterials offering a direct specification of permittivity and permeability of LH MTM and much faster analysis than analog implementation of the MTM transmission line networks on circuit simulator. If the case of differential numerical techniques in the time-domain, such as Finite-Difference Time-Domain (FD-TD) method [3] and Transmission-Line Matrix (TLM) method [4] as most popular, such numerical model, if developed, can allow for time-harmonic and transient simulation of MTM and analysis of their dispersive behaviour for different excitation functions.

As far as FD-TD method is concerned, several techniques have been already developed to incorporate frequency dispersion of MTM, some of them explained and referenced in [3]. TLM method is a numerical network model of Maxwell's field equations and therefore perfectly suited to realization of metamaterials as host transmission lines with embedded lumped series capacitors and shunt inductors [2]. Following this loaded transmission line approach of metamaterials, TLM method has been enhanced in [5] with a model based on insertion of reactive periodic elements into the conventional TLM mesh of so-called link lines. However, in contrast to FDTD method, that numerical TLM model allowed the direct specification of MTM properties only at one design frequency, for which the embedded reactive stubs were calculated.

In order to develop TLM model that will capture dispersive behaviour of LH MTM in a wide frequency range, TLM method based on Z-transforms techniques is applied in this paper. This approach is fully verified for accurate time-domain description of general frequency-dependent properties in isotropic, bi-isotropic, anisotropic and nonlinear materials that can be found in nature [6,7]. Such enhanced TLM method is extended in the paper by the model based on Drude function for describing frequency dependant behaviour of LH MTM permittivity and permeability and bilinear transformation technique to transfer this dependence in the time domain. Model is incorporated into three-dimensional (3D) TLM mesh and implemented in 3D TLM*scn*-Z software, designed at the Microwave Lab at the Faculty of Electronic Engineering in Nis. This software has been already successfully applied to resonant frequency calculation of metallic cavity loaded with dielectric slab exhibiting frequency-dependent complex permittivity [8]. Accuracy and efficiency of dispersive TLM model of LH MTM are verified on some characteristic examples in comparison with analytical solutions.

II. DISPERSIVE 3D TLM MODEL OF LH MTM

In [6,7], the scattering procedures for various types of frequency-dependent so-called right-handed materials (i.e. natural materials) have been developed. Exponential Z-transform technique was mainly used to incorporate right-handed material EM properties into 1D and 3D TLM algorithm. Main task was to determine, by using an appropriate conductivity or susceptibility model, elements of some of fraction expansions forms given below:

$$(1+z^{-1})g_e(z) = g_{e0} + z^{-1}(g_{e1} + \overline{g_e(z)}) \quad (1)$$

$$(1+z^{-1})r_m(z) = r_{m0} + z^{-1}(r_{m1} + \overline{r_m(z)}) \quad (2)$$

$$(1-z^{-1})\chi_{e,m}(z) = \chi_{e,m0} - z^{-1}(\chi_{e,m1} + \overline{\chi_{e,m}(z)}) \quad (3)$$

where: g_e is normalized electric conductivity, r_m is normalized magnetic resistivity, χ_e is electric susceptibility and χ_m is magnetic susceptibility.

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Realistic LH MTM can be characterized by using either the Lorentz or Drude dispersion function which differs in term of frequency region bandwidth in which real parts of EM properties are negative. In this paper the Drude function for both the permittivity $\varepsilon(\omega)$ and permeability $\mu(\omega)$ with identical dispersion forms [3] is used to find partial fraction expansions forms elements:

$$\varepsilon(\omega) = \varepsilon_0 \left(\varepsilon_\infty - \frac{\omega_{pe}^2}{\omega^2 - j\omega\gamma_e} \right) \tag{4}$$

$$\mu(\omega) = \mu_0 \left(\mu_\infty - \frac{\omega_{pm}^2}{\omega^2 - j\omega\gamma_m} \right) \tag{5}$$

where the Drude function parameters $\omega_{pe,m}$ and $\gamma_{e,m}$ represent the electric or magnetic plasma frequency and the corresponding collision frequency, respectively.

In this paper, y component of electric field and z component of magnetic field will be used as an illustration how to incorporate the Drude function of permittivity and permeability into 3D TLM Z-Transform method algorithm following notation used in [6]. Similar expression can be obtained for other electric and magnetic field components. Also, it is assumed that normalized electric conductivity and magnetic resistivity, g_e and r_m, are frequency independent. From Eqs. (4) and (5) follows that electric and magnetic susceptibility of LH MTM have a form in the **s**-domain as:

$$\chi_e(s) = \chi_{e\infty} + \frac{\omega_{pe}^2}{\gamma_e} \left(\frac{1}{s} - \frac{1}{s + \gamma_e} \right) \tag{6}$$

$$\chi_m(s) = \chi_{m\infty} + \frac{\omega_{pm}^2}{\gamma_m} \left(\frac{1}{s} - \frac{1}{s + \gamma_m} \right) \tag{7}$$

Applying rather bilinear Z-transform technique s→2(1-z^{-1})/[*dt*(1+z^{-1})] to Eqs. (6) and (7), instead of exponential Z-transform, as produces much better results, electric and magnetic susceptibility can be represented in the **z**-domain as:

$$\chi_e(z) = \chi_{e\infty} - K_e(1 + z^{-1}) \left(\frac{1}{1 - z^{-1}} - \frac{1}{B_e(1 - z^{-1}A_e / B_e)} \right) \tag{8}$$

$$\chi_m(z) = \chi_{m\infty} - K_m(1 + z^{-1}) \left(\frac{1}{1 - z^{-1}} - \frac{1}{B_m(1 - z^{-1}A_m / B_m)} \right) \tag{9}$$

where the coefficients: $K_{e,m} = -\omega_{pe,m}^2 dt / (2\gamma_{e,m})$, $B_{e,m} = 1 + \gamma_{e,m} dt / 2$ and $A_{e,m} = 1 - \gamma_{e,m} dt / 2$.

Taking partial fraction expansions to represent frequency dependence of electric and magnetic susceptibility as function of the value at the previous time-step leads to:

$$\begin{aligned}(1 - z^{-1})\chi_e(z) &= \chi_{e\infty} - K_e + K_e / B_e - \\ &z^{-1} \left(\chi_{e\infty} + K_e - a_{1e}K_e / B_e + z^{-1} \frac{b_{1e} / 4}{1 - z^{-1}a_{1e}} \right)\end{aligned} \tag{10}$$

$$\begin{aligned}(1 - z^{-1})\chi_m(z) &= \chi_{m\infty} - K_m + K_m / B_m - \\ &z^{-1} \left(\chi_{m\infty} + K_m - a_{1m}K_m / B_m + z^{-1} \frac{b_{1m} / 4}{1 - z^{-1}a_{1m}} \right)\end{aligned} \tag{11}$$

and with their comparison with Eq. (3), elements of partial fraction expansions forms can be found as:

$$\chi_{e,m0} = \chi_{e,m\infty} - K_{e,m} + K_{e,m} / B_{e,m} \tag{12}$$

$$\chi_{e,m1} = \chi_{e,m\infty} + K_{e,m} - a_{1e,m}K_{e,m} / B_{e,m} \tag{13}$$

$$\overline{\chi_{e,m}(z)} = z^{-1} \frac{b_{1e,m} / 4}{1 - z^{-1}a_{1e,m}} \tag{14}$$

where the coefficients: $a_{1e,m} = A_{e,m} / B_{e,m}$ and $b_{1e,m} / 4 = K_{e,m} 4\gamma_{e,m} dt / [2B_{e,m}^3]$.

Update scheme for y component of electric and z component of magnetic field includes additional accumulators S_{ed} and S_{md}, respectively, that can be evaluated by using state variables defines as: $X_{1e} = z^{-1}V_y / (1 - z^{-1}a_{1e})$ and $X_{1m} = z^{-1}i_z / (1 - z^{-1}a_{1m})$:

$$S_{ed} = 4\overline{\chi_e(z)}V_y = b_{1e}X_{1e} \tag{15}$$

$$S_{md} = 4\overline{\chi_m(z)}i_z = b_{1m}X_{1m} \tag{16}$$

Finally, the complete key step in scattering calculation related to y component of electric and z component of magnetic field, with a frequency independent normalized electric conductivity and magnetic resistivity, g_e and r_m, can be represented as:

$$V_y = T_e(2V_y^r + z^{-1}S_{ey}), \quad i_z = T_m(-2i_z^r + z^{-1}S_{mz}) \tag{17}$$

$$S_{ey} = 2V_y^r + k_eV_y + b_{1e}X_{1e}, \quad S_{mz} = -2i_z^r + k_m i_z + b_{1m}X_{1m} \tag{18}$$

$$X_{1e} = z^{-1}a_{1e}X_{1e} + z^{-1}V_y, \quad X_{1m} = z^{-1}a_{1m}X_{1m} + z^{-1}i_z \tag{19}$$

with the coefficients: $T_e = (4 + g_e + 4\chi_{e0})^{-1}$, $T_m = (4 + r_m + 4\chi_{m0})^{-1}$, $k_e = -(4 + g_e - 4\chi_{e1})$ and $k_m = -(4 + r_m - 4\chi_{m1})$.

III. NUMERICAL ANALYSIS

Accuracy of developed dispersive 3D TLM model of metamaterials is verified on the example of uniform 10-GHz TEM wave propagation through metamaterial slab inserted between two air slabs. Problem geometry is shown in Fig.1.

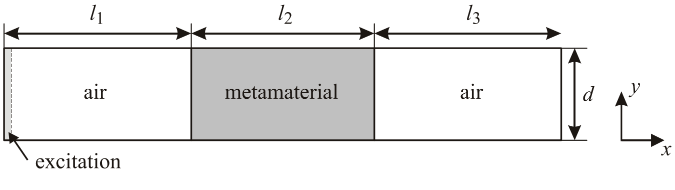

Fig.1 Metamaterial slab inserted between two air slabs

The length of each slab is $l_1= l_2= l_3=70$ mm and width is $d=10$ mm. Parameters of the Drude function are chosen so that at 10 GHz, the refractive index of the LH MTM is $n=-2$ ($\varepsilon_r = \mu_r = -2$ at 10 GHz) and its characteristic impedance is 377 Ω. Lossless case is considered ($\omega_{pe,m} >> \gamma_{e,m}$). Excitation is in the form of monochromatic 10-GHz wave with electric field polarized in *z* direction. For modelling of this basically two-dimensional structure, uniform TLM mesh with 210×10×1 nodes is used. Electric and magnetic walls, as boundary conditions, are applied in *z* and *y* planes, respectively, in order to support TEM wave propagation. Electric field distribution at times t_1 and t_2, separated by 10*dt* interval, is shown in Fig.2 where it can be seen that the wave impedance of LH MTM slab is matched to air, its phase velocity is negative and, as its wavelength, is half than in air.

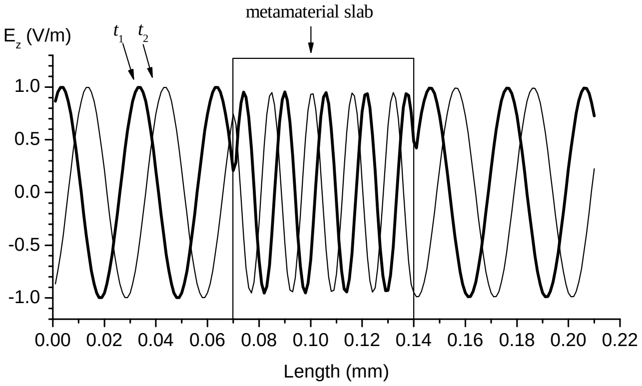

Fig.2. Electric field distribution, $t_2 - t_1 = 10dt$

Capability of dispersive 3D TLM model to account for frequency dispersive behaviour of LH MTM is illustrated on the example of reflection coefficient calculation of air-metamaterial interface in a wide frequency range. Parameters of the Drude function are chosen differently for permittivity and permeability of LH MTM so that for an example $\varepsilon_r = -2$ and $\mu_r = -1$ at 10 GHz, giving the reflection coefficient magnitude of 0.172. Frequency dependance of real part of relative permittivity and permeability of LH MTM is shown in Fig.3.

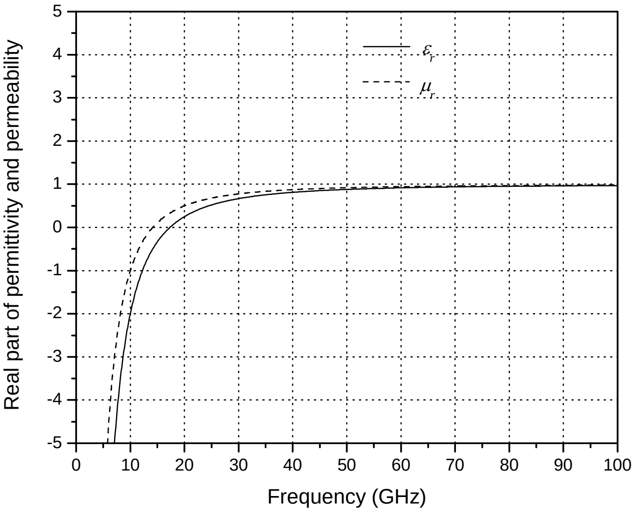

Fig.3: Real part of relative permittivity and permeability of LH MTM versus frequency

Initial pulse of Gaussian form is used to inject energy in the considered frequency range. The numerical procedure for wide bandwidth reflection coefficient calculation of air-metamaterial interface was performed in two TLM time domain simulation steps. At first step, the entire problem space was filled with air and electric field strength versus time, at a position one cell in front of the place where air-water interface should be placed, gave the incident field. A second simulation was performed when air-metamaterial interface existed, applying the dispersive TLM model of LH MTM properties. The result of the second simulation provided the total field at a position one cell in front of the interface. The reflected field was obtained by subtracting the incident field result (no metamaterial) from the total field result with the metamaterial interface present. The incident and reflected electric field versus time are shown in Fig.4.

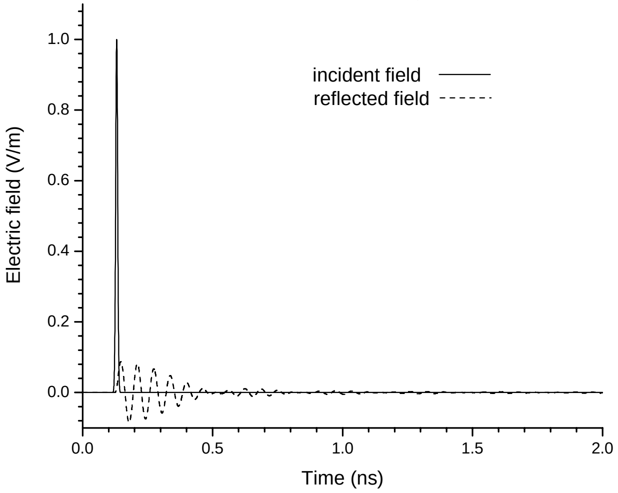

Fig.4 The incident and reflected electric field, at a position one cell in front of the air-metamaterial interface, versus time

The incident and reflected electric field data in the time domain were transformed to the frequency domain via discrete Fourier transform. The reflection coefficient magnitude at each frequency, calculated by dividing the transforms of the reflected field and the incident field, is in very good agreement with the analytic solution (Fig.5).

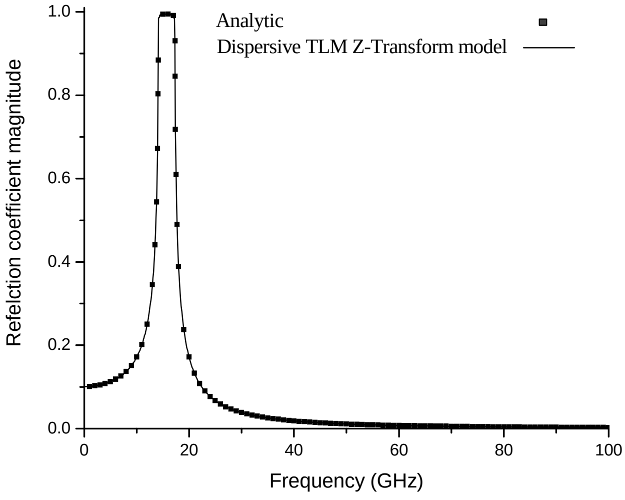

Fig.5 Reflection coefficient magnitude of air-metamaterial interface versus frequency

IV. CONCLUSION

3D dispersive TLM model of metamaterials based on Z-Transform of the Drude function, that describes EM poperites of LH MTM, has been derived and implemented. Accuracy of the model to account for frequency dispersive LH MTM behaviour is illustrated on the several examples which confirmed the theoretically predicted behaviour of metamaterials and reproduced the results of an excellent agreement with analytical solutions.

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