Algorithm for Fast Complex Hadamard Transform

Rumen P. Mironov[1], Roumen K. Kountchev[2]

***Abstract* – An algorithm for fast complex Hadamard transform is presented. The complex Hadamard matrices are factorized with set of sparse matrices on the base of classical Cooley-Tukey algorithm and obtained signal flow graphs of order 4, 8 and 16 are shown. The developed algorithm is simulated on Matlab 6.5 environment and the computational complexity is calculated.**

***Keywords* – digital signal processing, Complex Hadamard Transform, fast orthogonal transforms.**

I. INTRODUCTION

Discrete orthogonal (unitary) transforms [1], [2] have found applications in many areas of N-dimensional signal processing, spectral analysis, pattern recognition, digital coding, computational mathematic and etc. Dimensionality reduction in computation is a major signal processing application. Stated simply, these transform coefficients that are small may be excluded from processing operations, such as filtering, without much loss in processing accuracy. The discrete integer Walsh Hadamard Transform (WHT) is a fairly simple orthogonal transform and is an example of a generalized class of Fourier transforms [3]. The idea of using complex, rather than integer transforms matrices for spectral processing, analysis and watermarking has been shown in [4], [5], [6] and [7]. From the Complex Hadamard Transform (CHT), several complex decisions diagrams are derived and analysis of more general CHT properties for 1D and 2D signals are investigated [8].

In this paper an algorithm for Fast Complex Hadamard Transform (FCHT) is developed, using factorization of basis CHT matrices by the sparse matrices and the signal flow graphs illustrating the computation of orders 4, 8 and 16 are shown. The obtained results are similar to factorization and flow graphs of real Fast Hadamard Transform (FHT), which leads to considerable decreasing of mathematical computations. The difference between FCHT and FHT is entirely into the last iteration, which include the all complex operations. Comparison with the presented in [9] fast conjugate symmetric sequence-ordered Complex Hadamard Transform is made and the results are given.

The developed FCHT algorithm is simulated on Matlab 6.5 environment and it requires $N.log_2 N$ additions or subtractions and $N/2$ complex operations in the last iteration.

II. MATHEMATICAL DESCRIPTION

The coefficients of Complex Hadamard Transform matrix $[CH_N]$ with dimension N by N can be represented by the following equations [7], [8]:

$$\left| \begin{array}{l} c(u,v) = j^{uv} s(u,v) \\ c^*(u,v) = (-j)^{uv} s(u,v) \end{array} \right. , \tag{1}$$

where: $N = 2^n$, $j = \sqrt{-1}$, u, v = 0,1,…2^n-1 and

$$s(u,v) = \begin{cases} 1 & \text{for } n = 2 \\ (-1)^{\sum_{r=3}^{n} \lfloor u/2^{r-1} \rfloor \lfloor v/2^{r-1} \rfloor} & \text{for } n = 3,4,5..... \end{cases} , \tag{2}$$

is the sign function. Here $\lfloor . \rfloor$ is an operator, which represents the integer part of the result, obtained after the division.

From the equations (1) and (2) the CHT basis matrix of order 2^n and complex conjugated matrix calculated for n=2 and u ,v = 1,2,3,4 are presented as follows:

$$[CH_4] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \quad [CH_4]^* = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}. \tag{3}$$

The basis complex Hadamard matrices of order 2^n (n>2) can be received as the Kroneker product of a number of identical "core" matrices of order 2^{n-1} in the following way:

$$[CH_{2^n}] = \begin{bmatrix} [CH_{2^{n-1}}] & [CH_{2^{n-1}}] \\ [CH_{2^{n-1}}] & -[CH_{2^{n-1}}] \end{bmatrix}. \tag{4}$$

Using the basic forward one-dimensional complex Hadamard transform for n=2 from the input signal vector $\vec{A} = [a_1, a_2, a_3, a_4]$, the output spectral vector - $\vec{B} = [b_1, b_2, b_3, b_4]$ is received by the equations [7]:

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} , \left| \begin{array}{l} b_1 = a_1 + a_2 + a_3 + a_4 \\ b_2 = a_1 + ja_2 - a_3 - ja_4 \\ b_3 = a_1 - a_2 + a_3 - a_4 \\ b_4 = a_1 - ja_2 - a_3 + ja_4 \end{array} \right. . \tag{5}$$

The complex Hadamard matrix $[CH_4]$ can be decomposed of series of two sparse matrixes of order 4:

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$$[CHI_4] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} [I_2] & [I_2] \\ [I_2] & -[I_2] \end{bmatrix}, \qquad (6)$$

$$[CHJ_4] = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & j \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -j \end{bmatrix}, \qquad (7)$$

where $[I_{2^n}]$ is the identity matrix and decomposition can be written by the following:

$$[CH_4] = [CHJ_4].[CHI_4]. \qquad (8)$$

The first matrix $[CHI_4]$ is presented with a first sub-graph and the second matrix $[CHJ_4]$ is presented with a second one in the generalized CHT signal flow graph, shown on Fig.1:

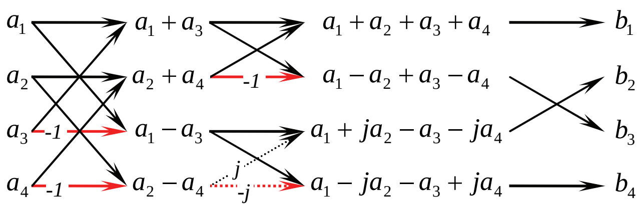

Fig. 1. Fast CHT signal flow graph of order 4.

The third sub-graph presents reordering of spectrum elements from equation (5). Multipliers are *+1* and *-1* as indicated by the black and red solid lines in real parts of sub-graphs and *+j* and *–j* as indicated by the black and red dashed lines in complex parts, respectively.

Using this approach the fast CHT algorithm and the corresponding signal flow graph of order 8 can be constructed in the following way:

- record the basic foreword complex Hadamard transform as linear system of 8 unknown values:

$$\left| \begin{aligned} b_1 &= a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 \\ b_2 &= a_1 + ja_2 - a_3 - ja_4 + a_5 + ja_6 - a_7 - ja_8 \\ b_3 &= a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + a_7 - a_8 \\ b_4 &= a_1 - ja_2 - a_3 + ja_4 + a_5 - ja_6 - a_7 + ja_8 \\ b_5 &= a_1 + a_2 + a_3 + a_4 - a_5 - a_6 - a_7 - a_8 \\ b_6 &= a_1 + ja_2 - a_3 - ja_4 - a_5 - ja_6 + a_7 + ja_8 \\ b_7 &= a_1 - a_2 + a_3 - a_4 - a_5 + a_6 - a_7 + a_8 \\ b_8 &= a_1 - ja_2 - a_3 + ja_4 - a_5 + ja_6 + a_7 - ja_8 \end{aligned} \right. \qquad (9)$$

- decomposition of complex Hadamard matrix $[CH_8]$ by the series of three sparse matrixes of order 8:

$$[CH_8] = [CHJ_8].[CHI_8]'.[CHI_8]'', \qquad (10)$$

$$[CHI_8]'' = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} [I_4] & [I_4] \\ [I_4] & -[I_4] \end{bmatrix} \qquad (11)$$

$$[CHI_8]' = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix} = \left[\begin{array}{c:c} [CHI_4] & 0 \\ \hdashline 0 & [CHI_4] \end{array}\right] \qquad (12)$$

$$[CHJ_8] = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & j & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -j & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & j \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -j \end{bmatrix} = \left[\begin{array}{c:c} [CHJ_4] & 0 \\ \hdashline 0 & [CHJ_4] \end{array}\right] \qquad (13)$$

where: $[CHI_8]''$, $[CHI_8]'$ and $[CHJ_8]$ are real and complex factorization sparse matrices;

- construction of signal flow graph for complex Hadamard transform of order 8, which is shown on Fig.2;

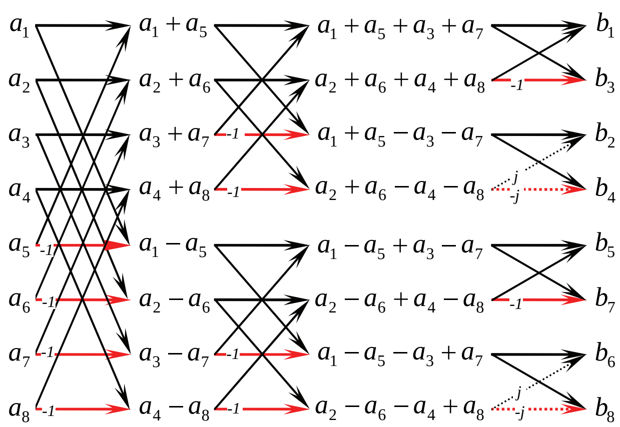

Fig. 2. Fast CHT signal flow graph of order 8.

Decompositions, shown in (8) and (10) can be generalized using Cooley-Tukey algorithm ([2], [3]) for the real [*CHI*] matrices factorization and multiplying with the complex matrix [*CHJ*]. The complex Hadamard matrix $[CH_N]$ of order $N=2^n$ can be presented by the equation:

$$[CH_N]=[CHJ_N].\prod_{r=1}^{n-1}[G_r(N)], \tag{14}$$

where: $n=\log_2 N$, $[G_r(N)]$ are the sparse matrices with two non-zero elements in each row, which have the following block-diagonal structure:

$$[G_r(N)]=\begin{bmatrix}[A(r)] & 0 & ..... & 0\\ 0 & [A(r)] & ..... & 0\\ ..... & ..... & ..... & ......\\ 0 & 0 & ..... & [A(r)]\end{bmatrix}. \tag{15}$$

The sub-matrices $[A(r)]$ are defined as Kroneker product of the matrices:

$$[A(r)]=[H_2]\otimes[I_{2^r}], \tag{16}$$

where: $[I_{2^r}]$ is identity matrix of size $2^r x 2^r$, and $[H_2]$ is real basic Hadamard matrix of order 2.

Using definitions in (7) and equations (14)-(16) the fast CHT for *N*-components vector $\vec{A}=[a_1,a_2,.....,a_{N-1},a_N]$ can be presented in the following way:

$$\vec{B}=[CHJ_N].[G_1(N)].[G_2(N)].....[G_{n-1}(N)].\vec{A}, \tag{17}$$

or as a sequence of elementary transformations:

$$\begin{aligned}\vec{C}_1 &= [G_{n-1}(N)].\vec{A}\\ \vec{C}_2 &= [G_{n-2}(N)].\vec{C}_1\\ &..........................\\ \vec{B} &= [CHJ_N].\vec{C}_{n-1}\end{aligned}, \tag{18}$$

where: $\vec{C}_1,\vec{C}_2,....\vec{C}_{n-1}$ is the sequence of "intermediated" vector-iterations, which are received by the transformations with sparse matrices.

As a sample, from equation (14) for $[CH_8]$ factorization of order *N*=8 are obtained:

$$[A(1)]=[H_2]\otimes[I_2];\ [A(2)]=[H_2]\otimes[I_4]$$

$$[CH_8]=[CHJ_8].\prod_{r=1}^{2}[G_r(8)]=[CHJ_8].[G_1(8)].[G_2(8)]$$

$$[G_1(8)]=\left[\begin{array}{cc:cc}[I_2] & [I_2] & & 0\\ [I_2] & -[I_2] & & \\ \hdashline & & [I_2] & [I_2]\\ 0 & & [I_2] & -[I_2]\end{array}\right]=\left[\begin{array}{c:c}[CHI_4] & 0\\ \hdashline 0 & [CHI_4]\end{array}\right]. \tag{19}$$

$$[G_2(8)]=\left[\begin{array}{c:c}[I_4] & [I_4]\\ \hdashline [I_4] & -[I_4]\end{array}\right]$$

The received from (19) equations are completely identical with the equations (10)-(13). The developed fast complex Hadamard transform (FCHT) algorithm can be illustrated by the signal flow graph, shown on Fig.2.

The described algorithm can be used for the reverse complex Hadamard transformation. The flow graphs are identical and all output components must be divided on the *N*.

III. Experimental Results

The developed FCHT algorithm is simulated on Matlab 6.5 environment. The basis complex Hadamard matrices are calculated using equation (4) and factorization sparse matrices using equation (14) for orders *N*=4, 8, 16, 32, 64, 128, 256 and 512 are constructed. The received results are identical, which indicates that proposed factorization is correct. The FCHT algorithm is similar with the real FHT algorithm, which leads to considerable decreasing of mathematical computations. The difference between FCHT and FHT is entirely into the last iteration, which includes all complex operations.

The developed FCHT algorithm requires $N.log_2 N$ additions or subtractions and *N/2* complex operations in the last iteration. The presented in [9] fast conjugate symmetric sequence-ordered Complex Hadamard Transform algorithm requires $N.log_2 N$ additions or subtractions, but the complex operations are 3 times more than FCHT.

As a sample the obtained signal flow graph of order 16 is given on the Fig.3.

Using the matrix descriptions for the 2D Complex Hadamard Transform in [8], the 2D FCHT algorithm can be realized by applying of 1D FCHT on the rows of the input image matrix and after then applying the 1D FCHT on the columns of the obtained matrix. The calculation complexity of 2D FCHT can be evaluated from the complexity of 1D FCHT and require: $2.N^2.log_2 N$ additions or subtractions and N^2 complex operations.

IV. Conclusion

A class of complex Hadamard matrix is presented. An algorithm for fast CHT matrices calculation is developed on the base of classical Cooley-Tukey algorithm. Additional factorization matrices for the complex calculations are constructed.

The main advantages of the developed FCHT algorithm are:

- minimizing of complex calculations;
- the input components of each iteration can be substituted with the resulting ones, which reduces the used memory only for calculated *N* components;
- the 2D FCHT algorithm is based on the described 1D FCHT algorithm and is calculated by the same way as a real 2D FHT.

The presented fast ordered Complex Hadamard Transform can be used in digital signal processing for spectral analysis, pattern recognition, digital watermarking, transformation,

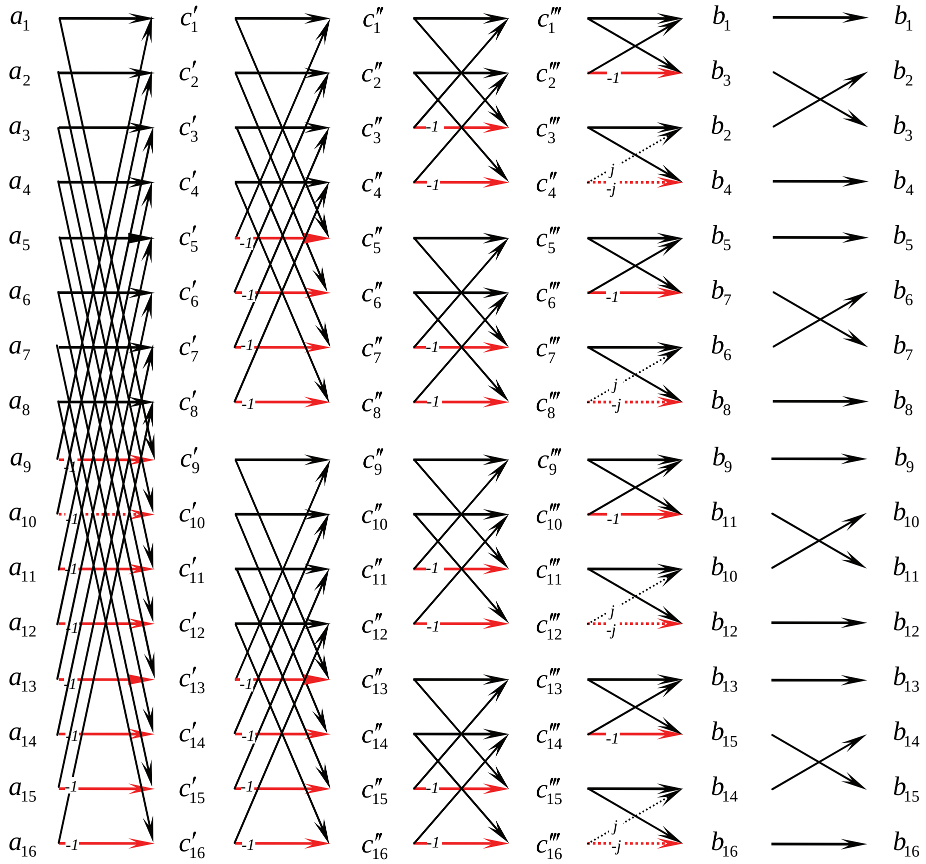

Fig. 3. Fast CHT signal flow graph of order 16.

coding and transmission of one-dimensional and two-dimensional signals.

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