Optimization of Polynomial Approximation in Series Based Software Direct Digital Synthesis of Signals

Borko G. Boyanov[1] and Mariana Iv. Shotova[2]

***Abstract* – In this paper an optimization of the software method for Direct Digital Synthesis of signals, based on series approximation of the sine wave is discussed. Three known sine-wave approximations are compared considering the spurious free dynamic range of the spectrum. An optimization of the polynomial approximation is proposed and discussed here. The signal synthesis requires reduced number of mathematical operations by taking advantage of the sine wave symmetry. Additional increase of the dynamic range is achieved by genetic algorithm optimization of the polynomial coefficients.**

***Keywords* – DDS, series, polynomial, optimization, GAO.**

I. INTRODUCTION

The Direct Digital Synthesis (DDS) is a technique for generating a high quality sine wave through a digitally defined frequency. The software implementation of DDS (SDDS) based on digital signal processor has two main versions – using ROM table of the sine wave, and series approximation of the sine wave.

The first one is most common and faster, however, due to restricted ROM table size, the spurious free dynamic range (SFDR) D of the spectrum of the synthesized signal, which is measure of quality, is also limited [1].

The advantage of the SDDS exploiting series approximation is elimination of the ROM table. A drawback however is the bigger number of the required mathematical operations, which results in lower sampling frequency.

There are several basic sine wave polynomial approximations, which can be used in DDS [2].

Analog Devices Inc. [3] suggested a 5^{-th} order polynomial approximation of the sine wave in the range [0, π/2]. The dynamic range D is around 105 dB. The number of operations is 5 multiplications and 4 addition/subtractions.

More recently, methods combining the use of smaller tables with the evaluation of a low-degree polynomials have been proposed [4], [5]. However some of them exhibit argument-dependent execution time, which is not acceptable in real word DDS.

The SDDS, proposed here, requires reduced number of mathematical operations by taking advantage of the sinus' symmetry – using approximation in the range [-π/2, π/2], thus eliminating the even-order components of the polynomial. This is implemented here using MATLAB and its ***polyfit*** function. The SDDS suggested here, is based on 7^{-th} order polynomial and features dynamic range of 128dB.

The complicated nature of the effect of the polynomial coefficients on the spectral spur levels is factor, which suggests utilization of generic algorithm optimization (GAO) of the coefficients, aimed at minimization of the spectral spur levels. The application of GAO to SDDS results in sets of coefficients of the 7^{-th} order polynomial, which increase the SFDR up to 133dB.

II. SINE WAVE SERIES AND POLINOMIALS

The series, which can be used for sine wave synthesis is [2]

$$\sin(\alpha) = \alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \frac{\alpha^7}{7!} + ... \qquad (1)$$

In this series, due to sinus' odd-symmetry, even-order components are excluded. To minimize the error due to discarded series' component, the argument range is limited to [0, π/2]. A plot of calculation error $e_{(\alpha)}$, over the range [0, 2π] when 4 components of the series are used, is shown in Fig.1 with solid line. The dashed line represents the discarded 9^{-th} order component $x_{9(\alpha)}$, which dominates the error.

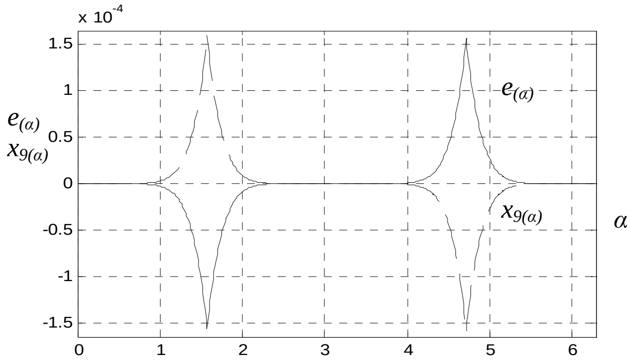

Fig.1. Calculation error $e_{(\alpha)}$ and 9^{-th} order component $x_{9(\alpha)}$.

The elimination of the high-order components of the series results in large error at angles close to odd multiples of π/2, while the error at angles close to multiples of π is very small.

In the case of the DDS, the error spectrum is of interest. An error signal, which contains 16 periods of error "wave" $e_{(\alpha)}$ is composed, and FFT is applied. The main part of the spectrum is shown in Fig.2. The spectral line at k = 16 is the fundamental frequency, while the other spectral lines represent odd-order harmonics. Since the amplitude of the synthesized signal is A = 1, (0 dB), the SFDR is defined by the level of the third harmonic at k = 48 $D = -L_3$ = 91.4dB.

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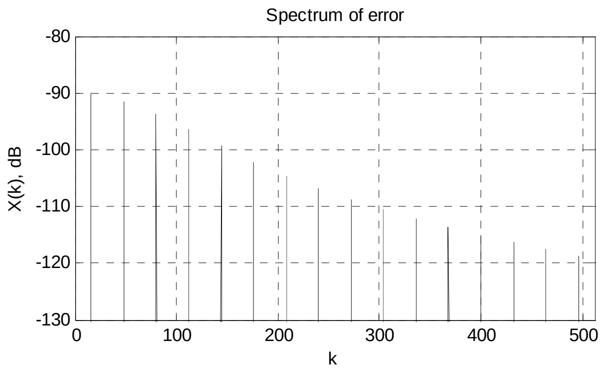

Fig.2. Main part of the FFT error spectrum.

For its 16-bit DSP Analog Devices Inc. suggested a 5th order polynomial approximation of the sine wave in the range [0, π/2]

$$\sin(\alpha) = 3.140625\beta + 0.02026367\beta^2 - 5.325196\beta^3 + 0.5446778\beta^4 + 1.800293\beta^5, \quad (2)$$

where $\beta = \alpha/\pi$, $0 \le \beta \le 0.5$.

Rearranging the series in the form, which is closer to DSP,

$$\sin(\alpha) = \beta(a_1 + \beta(a_2 + \beta(a_3 + \beta(a_4 + \beta a_5)))), \quad (3)$$

the number of the required operations is 5 multiplications and 4 addition/subtractions.

A plot of calculation error $e_{(\alpha)}$, over the range [0, 2π] with this approximation is shown in Fig.3.

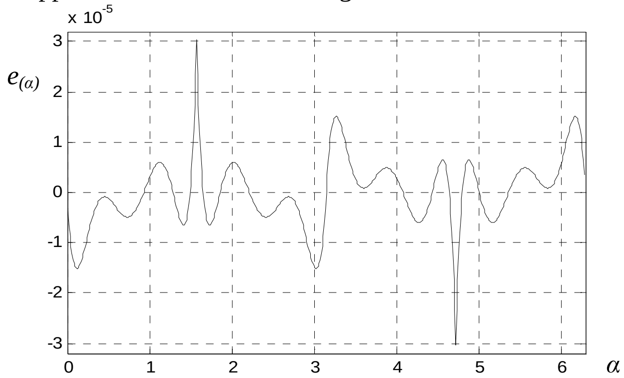

Fig.3. Sine wave error with ADI polynomial.

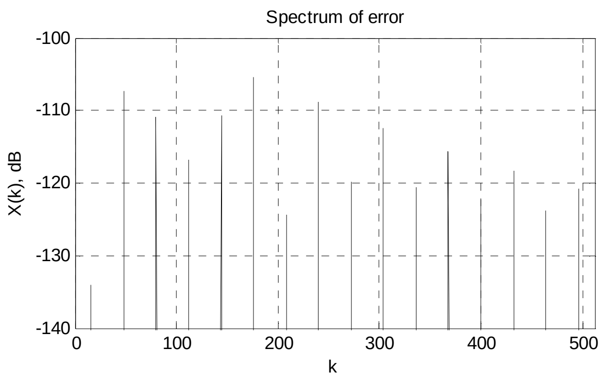

Fig.4. Main part of the error spectrum with ADI polynomial.

The main part of the spectrum is shown in Fig.4. The SFDR is defined by the level of the eleventh harmonic at k = 176 $D = -L_{11}$ = 105dB.

III. Improvement to the polinomial approximation

A drawback of the approximation (2) is that it does not take into account the sinus' odd symmetry. The approximation suggested here in the range [-π/2, π/2] eliminates the even-order components of the polynomial. This is implemented by MATLAB's *polyfit* function, which minimizes the root-mean-square error. The 7-th order polynomial obtained by this function is

$$\sin(\alpha) = a_1\alpha + a_3\alpha^3 + a_5\alpha^5 + a_7\alpha^7, \quad (4)$$

where $a_1 = 0.9999974$, $a_3 = -0.1666513$, $a_5 = 0.0083092$, and $a_7 = -0.00018437$.

A plot of the error $e_{(\alpha)}$, over the range [-π/2, 3π/2] with this approximation is shown in Fig.5.

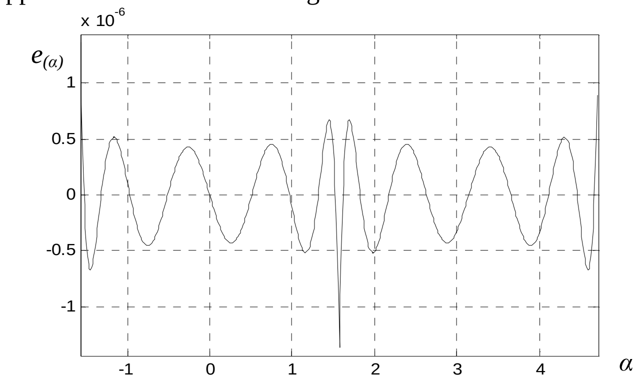

Fig.5. Sine wave error with 7th order polynomial.

The rearranged form of the polynomial is

$$\sin(\alpha) = \alpha(a_1 + \alpha^2(a_3 + \alpha^2(a_5 + a_7\alpha^2))), \quad (5)$$

where α^2 is calculated in advance. With this in mind the number of the required mathematic operations is 5 multiplications and 3 addition/subtractions.

The main part of the spectrum is shown in Fig.6.

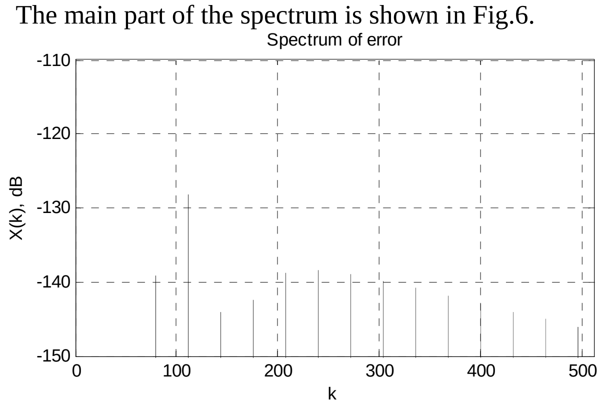

Fig.6. Error spectrum with 7th order polynomial.

The SFDR is defined by the level of the seventh harmonic at k = 112 , $D = -L_7$ = 128dB. Thus, exploiting the sinus' odd symmetry, the number of the polynomial components is reduced, while the dynamic range of the DDS is increased.

The seventh harmonic dominates the spectrum, and well the error in the time domain in Fig.5.

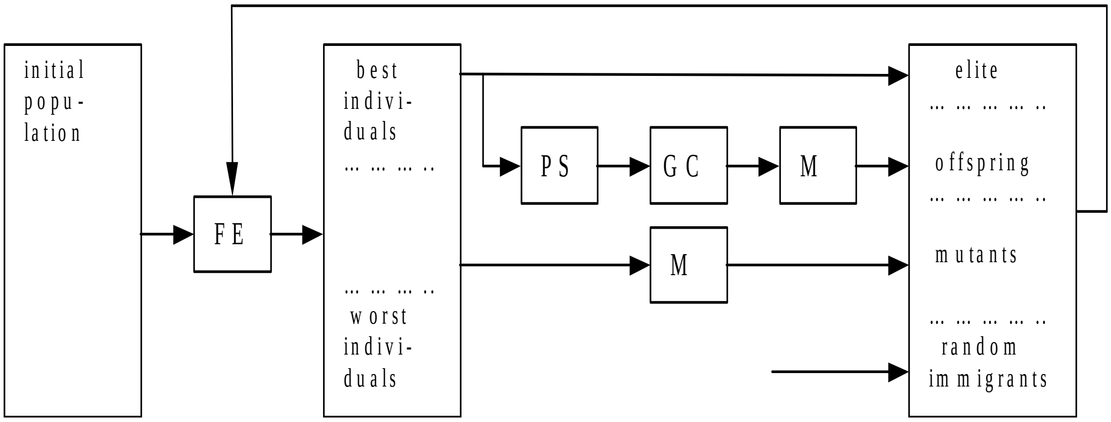

Fig.7. Generalized block diagram of a genetic algorithm.

IV. Genetic optimization of the polinomial

A drawback of the *polyfit* function, from DDS point of view, is that the approximation is based on minimization of the root-mean-square error, while for DDS it is important to minimize the spectral spur levels.

The complicated nature of the effect of the polynomial coefficients on the levels of the harmonics is factor that suggests the utilization of generic algorithm. This is a relatively new optimization method, which provides stochastic search over the parameter space, guided by fitness evaluation towards specific goal. Although relatively slow, GAO can handle complex optimization problems, especially when the goal is to find near-best extreme in multimodal function domain. GAO also features global search from many parameter space points, capability to escape from local extreme [6], [7].

Generalized block diagram of a genetic algorithm is shown in Fig.7. Usually, GAO comprises three operations - Selection, Genetic operations, and Replacement, which are repeated in genetic cycle. As in nature, population consists of individuals, each represented by its chromosome. The chromosome encodes all parameters of an individual. The genetic cycle starts with evaluation of the fitness (FE) of each individual. Then the individuals are ranked, accordingly to their fitness values, and a particular group of best individuals (parents) is selected by operation, called parent selection (PS), to generate Offspring. Genetic operators as Genetic crossover (GC) and Mutation (M) are applied then to produce offspring. Exploiting replacement strategy individuals of the current population are replaced and thus a new population is composed. Such genetic cycle is repeated until certain termination criterion is reached (predefined number of genetic cycles, or fitness threshold).

Binary coding of the coefficients and chromosome

There are four coefficients a_i to be encoded. In order to facilitate the flexibility in exploring the parameter space, the following mapping is exploited

$$a_i = a_{i0} - \Delta_i / 2 + \delta_i \sum_{j=0}^{b-1} b_{ij} 2^j \,, \quad i = 1,3,5,7. \qquad (6)$$

where a_{i0} are the initial values, calculated by *polyfit* function, Δ_i are the ranges of the variation, b_{ij} are the bits, b is the length of the binary string for each coefficient, and δ_i are the quantizing steps. These parameters are predefined for each coefficient Δ_1 = 0.00002, Δ_2 = 0.0002, Δ_3 = 0.0003, Δ_4 = 0.0020, $\delta_i = \Delta_i\, 2^{-b}$. In the developed GAO gene length b = 12 is chosen, which defines chromosome length L = 48 bits, and a search space of 2^{48} points.

Fitness evaluation

Since the objective of GAO is to reduce the spectral spur levels, the error signal and its spectrum are to be calculated. For each individual of the population the chromosome is split into genes and, accordingly to the mapping scheme (6), the coefficients are calculated. This set of coefficients is passed to the error and spectrum calculation function, which returns the levels of spectral spurs in decibels.

Parent selection and genetic operations

The number of the best individuals, selected for reproduction, is N_b = 30. These parents produce 60 child chromosomes by single-point crossover and double mutation with probability p_m = 0.995. Mixed replacement strategy is exploited: 30 best chromosomes are directly copied into the new population, which represents elitism, 210 mid-rank chromosomes undergo mutations and 120 random immigrants replace the worst chromosomes.

Results of the GA optimization

Tens of runs of the dedicated GAO program in MATLAB were performed, several of which resulted in sets of polynomial coefficients ensuring SFDR $D \approx$ 133dB. One such set is a_1=0.999997489622539, a_3=-0.166651585015804, a_5=0.008310170923798, and a_7 = -0.000184768956791.

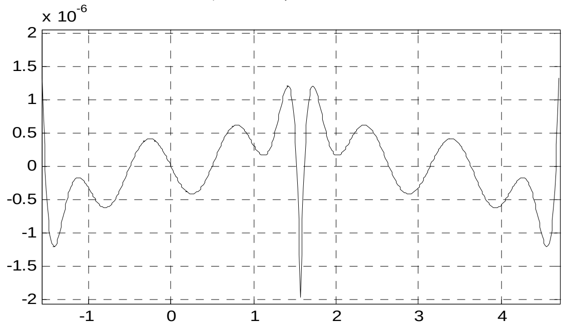

Fig.8. Sine wave error with GAO polynomial.

A plot of the error $e_{(\alpha)}$ over the range [-π/2, 3π/2] with these coefficients is shown in Fig.8. The shape of the error, excluding the fundamental component, is more irregular, compared to Fig.5, spreading the energy over the spectrum.

The main part of the error spectrum is shown in Fig.9. The SFDR is defined by the seventh harmonic at k = 112, $D = -L_7$ = 133dB. The strongest component at k = 16 indicates deviation of the amplitude of the fundamental component from the nominal value **A**=1. Such an inaccuracy of the magnitude (relative value $\approx 5 \cdot 10^{-7}$) is negligible in practice.

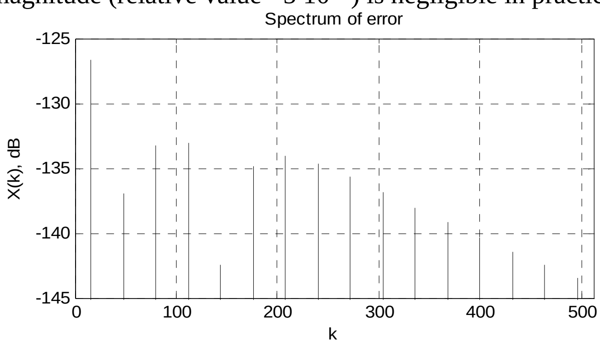

Fig.9. Error spectrum with GAO polynomial coefficients.

The development of the GA optimization over the time is illustrated in Fig.9, where the decrease of the maximal spur level L_s is shown.

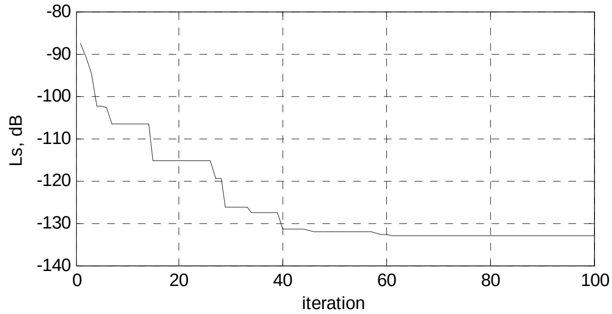

Fig.9. Development of the GAO over the time (iterations).

A distribution of the fitness of the individuals within the population at the end of the GAO is shown in Fig.10. About 30 of the individuals (the elite) have fitness close to the best one (-133dB).

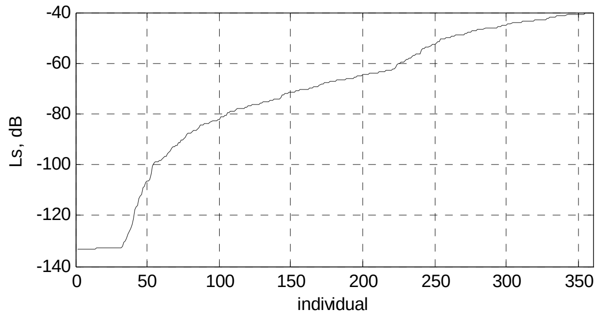

Fig.10. Fitness of the population at the end of GAO.

The GA optimization was performed on desktop machine with AMD Athlon 64X2 DualCore 5200+ 2.7GHz processor. Single run with specified GAO parameters and 1000 iterations takes approximately 80 s.

V. CONCLUSION

The optimization of the polynomial approximation of the sine wave aimed at Direct Digital Synthesis of sine wave signal was considered. Taking into account the symmetry of the sine wave, and applying GAO, minimizing the spectral spur levels, a polynomial of 7^{-th} order, which increases the spurious free dynamic range D up to 133 dB is suggested.

Further research, relevant to the problem, may focus on the following topics:

- polynomial approximation with polynomial of other order;
- quantizing effects in 16- and 32-bit implementations;
- fusing the GAO and the Gradient based search.

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