Applying of the Algorithm of Lagrange Multipliers in the Removal of Blur in Images

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***Abstract* – This paper presents a non-iterative method that finds application in a broad scientific field such as digital image restoration. The method, based on algorithm of Lagrange multipliers, is use for the removal of blur in an X-ray caused by uniform linear motion. This method assumes that linear motion corresponds to an integral number of pixels. The resolution of the restored image remains at very high level. The main contribution of the method was found on the Improvement in Signal to Noise Ration (*ISNR*) that has been increase significantly compared to the classic techniques.**

***Keywords* – Image restoration, Lagrange multipliers, matrix equation, X-ray.**

I. INTRODUCTION

Images are produced to record or display useful information. Due to imperfections in the imaging and capturing process, however, the recorded image invariably represents a degraded version of the original scene. The undoing of these imperfections is crucial to many of the subsequent image processing tasks. There exists a wide range of different degradations that need to be taken into account, covering for instance noise, geometrical degradations (pin cushion distortion), illumination and color imperfections (under/ overexposure, saturation), and blur [1].

Blurring is a form of bandwidth reduction of an ideal image owing to the imperfect image formation process. It can be caused by relative motion between the camera and the original scene, or by an optical system that is out of focus. When aerial photographs are produced for remote sensing purposes, blurs are introduced by atmospheric turbulence, aberrations in the optical system, and relative motion between the camera and the ground. Such blurring is not confined to optical images; for example, electron micrographs are corrupted by spherical aberrations of the electron lenses, and Computed tomography scans suffer from X-ray scatter.

The field of image restoration (sometimes referred to as image deblurring or image deconvolution) is concerned with the reconstruction or estimation of the uncorrupted image from a blurred one. Essentially, it tries to perform an operation on the image that is the inverse of the imperfections in the image formation system. In the use of image restoration methods, the characteristics of the degrading system are assumed to be known a priori.

The paper concentrates on using algorithm of Lagrange multipliers and other different methods for removing the removal of blur in an X-ray caused by uniform linear motion. We assume that the linear motion corresponds to an integral number of pixels and is aligned with the horizontal (or vertical) sampling.

For comparison, we used two commonly used filters from the collection of least-squares filters, namely Wiener filter and the constrained least-squares filter [2]. Also we used in comparison the iterative nonlinear restoration based on the Lucy-Richardson algorithm [3], [4].

This paper is organized as follows. In the second section we present process of image formation, problem formulation and the method for reconstruction of the blurred image. We observe certain enhancement in the Improvement in Signal to Noise Ration (*ISNR*) compared with other standard methods for image restoration, which is confirmed by the numerical examples reported in the last section.

II. METHOD FOR RECONSTRUCTION OF THE BLURRED IMAGE

A. Image Formation

We assume that the blurring function acts as a convolution kernel or point-spread function $h(n_1,n_2)$ and the image restoration methods that are described here fall under the class of linear spatially invariant restoration filters. It is also assumed that the statistical properties (mean and correlation function) of the image do not change spatially. Under these conditions the restoration process can be carried out by means of a linear filter of which the point-spread function (PSF) is spatially invariant, i.e., is constant throughout the image. These modeling assumptions can be mathematically formulated as follows. If we denote by $f(n_1,n_2)$ the desired ideal spatially discrete image that does not contain any blur or noise, then the recorded image $g(n_1,n_2)$ is modeled as [2]:

$$g(n_1,n_2) = h(n_1,n_2) * f(n_1,n_2) = \sum_{k_1=0}^{N-1}\sum_{k_2=0}^{M-1} d(k_1,k_2)\, f(n_1-k_1, n_2-k_2). \quad (1)$$

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The objective of the image restoration is to make an estimate $f(n_1, n_2)$ of the ideal image, under the assumption that only the degraded image $g(n_1, n_2)$ and the blurring function $h(n_1, n_2)$ are given.

B. Problem Formulation

The problem can be summarized as follows: let H be a $m \times n$ real matrix. Equations of the form:

$$g = Hf, g \in \Re^m; f \in \Re^n; H \in \Re^{m \times n}, \tag{2}$$

describe an underdetermined system of m simultaneous equations (one for each element of vector g) and $n = m + l - 1$ unknowns (one for each element of vector f). Where the index l indicates linear motion blur in pixels and $n = m + l - 1$.

The problem of restoring an X-ray that has been blurred by uniform linear motion, usually results of camera panning or fast object motion can be expressed as, consists of solving the underdetermined system (2). A blurred image can be expressed as:

$$\begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ \mathrm{M} \\ g_n \end{bmatrix} = \begin{bmatrix} h_1 & \Lambda & h_l & 0 & 0 & 0 & 0 \\ 0 & h_1 & \Lambda & h_l & 0 & 0 & 0 \\ 0 & 0 & h_1 & \Lambda & h_l & 0 & 0 \\ \mathrm{M} & \mathrm{M} & \mathrm{M} & \mathrm{M} & \mathrm{M} & \mathrm{M} & \mathrm{M} \\ 0 & 0 & 0 & \Lambda & h_1 & \Lambda & h_l \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \mathrm{M} \\ f_m \end{bmatrix} \tag{3}$$

The element of matrix H are defined as: $h_i = 1/l$ for i=1, 2,..., l.

The objective is to estimate an original row per row f (contained in the vector f^T), given each row of a blurred g (contained in the vector g^T) and a priori knowledge of the degradation phenomenon H. We define the matrix F as the deterministic original X-ray, its picture elements are F_{ij} for i=1,…, r and for j=1,…, n, the matrix G as the simulated blurred can be calculated as follows:

$$G_{ij} = \frac{1}{l} \sum_{k=0}^{l-1} F_{i,j+k} \quad \text{for } i=1,\dots, r, j=1,\dots, \mathrm{m} \tag{4}$$

with $n = m + l - 1$, where l is the linear motion blur in pixels. Equation (4) can be written in matrix form as:

$$G = \left(HF^T\right)^T = FH^T. \tag{5}$$

Since there is an infinite number of exact solutions for f or F in the sense that satisfy the equation $g = Hf$ or $G = FH^T$, an additional criterion that find a sharp restored matrix is required.

C. Method for Reconstruction Based on Lagrange Multipliers

Solution also is defined as the vector in the solution space of the underdetermined system $g = Hf$ (2) whose first n components has the minimum distance to the measured data, i.e. $\left\| \hat{f} - g \right\| \to \min$, where $\hat{f}$ are the first n elements of f. We can express vector $\hat{f}$ as $\hat{f} = Pf$, with P a $m \times n$ matrix which projects the vector f on the support of g [5], [6]:

$$P = \begin{bmatrix} I_m & \left| \begin{matrix} 0 & \Lambda & 0 \\ 0 & \Lambda & 0 \\ \mathrm{M} & \mathrm{M} & \mathrm{M} \\ 0 & \Lambda & 0 \end{matrix} \right. \end{bmatrix} \tag{6}$$

The original optimization problem is now:

$$find \min_f \left\| Pf - g \right\|^2, \tag{7}$$

subject to the constraint $\left\| Hf - g \right\|^2 = 0$. Applying the technique of *Lagrange multipliers*, this problem can be alternatively formulated as an optimization problem without constraints:

$$V(f) = \lambda \left\| Hf - g \right\|^2 + \left\| Pf - g \right\|^2 \to \min \tag{8}$$

if λ is large enough. The solution of this problem is easy computing the partial derivative of criterion V respect to the unknown f:

$$\begin{aligned} &\frac{\partial}{\partial f} V(f) = 2\lambda H^T (Hf - g) + 2P^T (Pf - g) = 0 \\ &\hat{f} = \left[\lambda H^T H + P^T P\right]^{-1} \left[\lambda H + P\right]^T g. \end{aligned} \tag{9}$$

Matrix form of the solution of (9) is:

$$\hat{F} = G\left[\lambda H + P\right]\left[\left(\lambda H^T H + P^T P\right)^{-1}\right]^T. \tag{10}$$

III. Experimental Results

In this section we have tested the method based on Lagrange multipliers of X-ray images and present numerical results and compare with two standard methods for image restoration called least-squares filters: Wiener filter and constrained least-squares filter and the iterative method called Lucy-Richardson algorithm.

The experiments have been performed using Matlab programming language on an Intel(R) Core(TM)2 Duo CPU T5800 @ 2.00 GHz 32-bit system with 2 GB of RAM memory running on the Windows Vista Business Operating System.

In image restoration the improvement in quality of the restored image over the recorded blurred one is measured by the signal-to-noise ratio (*SNR*) improvement. The *SNR* of the recorded (blurred and noisy) image is defined as follows in decibels [7]:

$$SNR_g = 10\log_{10}\left(\frac{\text{Variance of } f(n_1,n_2)}{\text{Variance of } g(n_1,n_2)-f(n_1,n_2)}\right)(\text{dB}). \quad (11)$$

The *SNR* of the restored image is similarly defined as:

$$SNR_{\hat{f}} = 10\log_{10}\left(\frac{\text{Variance of } f(n_1,n_2)}{\text{Variance of } \hat{f}(n_1,n_2)-f(n_1,n_2)}\right)(\text{dB}). \quad (12)$$

Then, the improvement in *SNR* is given by:

$$ISNR = SNR_{\hat{f}} - SNR_g = 10\log_{10}\left(\frac{\text{Variance of } g(n_1,n_2)-f(n_1,n_2)}{\text{Variance of } \hat{f}(n_1,n_2)-f(n_1,n_2)}\right)(\text{dB}). \quad (13)$$

The improvement in *SNR* is basically a measure that expresses the reduction of disagreement with the ideal image when comparing the distorted and restored image. Note that all of the above signal-to-noise measures can only be computed in case the ideal image is available, i.e., in an experimental setup or in a design phase of the restoration algorithm.

The simplest and most widely used full-reference quality metric is the mean squared error (*MSE*) [8], computed by averaging the squared intensity differences of restored and reference image pixels, along with the related quantity of peak signal-to-noise ratio (*PSNR*). These are appealing because they are simple to calculate, have clear physical meanings, and are mathematically convenient in the context of optimization. The advantages of *MSE* and *PSNR* are that they are very fast and easy to implement. However, they simply and objectively quantify the error signal. With *PSNR* greater values indicate greater image similarity, while with *MSE* greater values indicate lower image similarity. Below *MSE*, *PSNR* are defined:

$$MSE = \frac{1}{rm}\sum_{i=1}^{r}\sum_{j=1}^{m}\left|f_{i,j}-\hat{f}_{i,j}\right|^2 \quad (14)$$

and,

$$PSNR = 20\log_{10}\left(\frac{MAX}{\sqrt{MSE}}\right)(\text{dB}), \quad (15)$$

where *MAX* is the maximum pixel value.

The X-ray image making provides a crucial method of diagnostic by using the image analysis. Fig. 1, Original Image, shows such a deterministic original X-ray image. Fig. 1, Degraded Image, presents the degraded X-ray image for *l=40*. Finally, from Fig. 1, Lagrange multipliers Image, Wiener Restored Image, Constrained LS Restored Image and Lucy-Richardson Restored Image, it is clearly seen that the details of the original image have been recovered. These figures demonstrate four different methods of restoration, the method of Lagrange multipliers, Wiener filter, Constrained least-squares (LS) filter, and Lucy-Richardson algorithm, respectively.

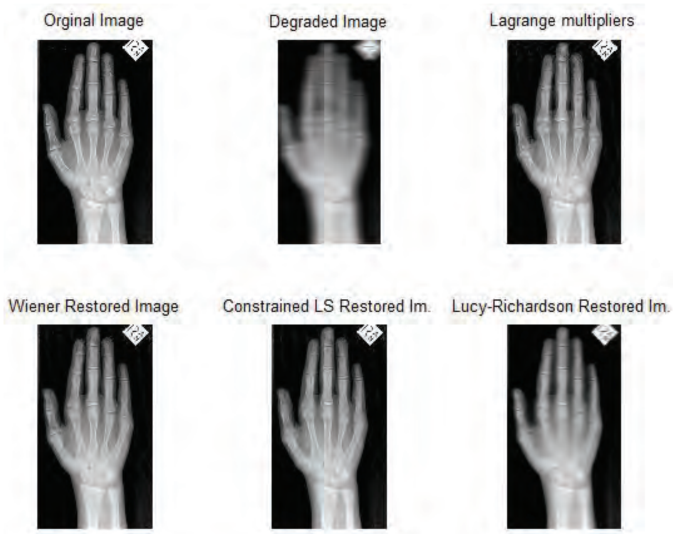

Fig. 1. Restoration in simulated degraded X-ray image for length of the blurring process, *l*=40

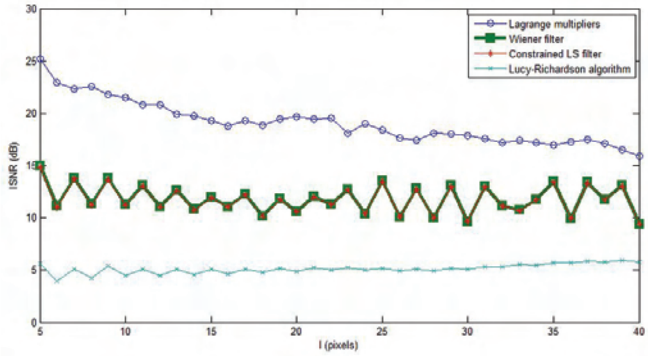

Fig. 2. Improvement in signal-to-noise-ratio vs. length of the blurring process in pixels

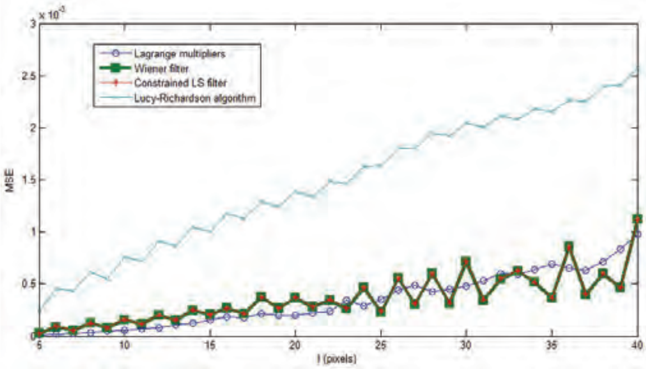

Fig. 3. Mean squared error vs. length of the blurring process in pixels

The difference in quality of restored images between the three methods is insignificant, and can hardly be seen by human eye. For this reason, the *ISNR* and *MSE* have been chosen in order to compare the restored images obtained by the proposed method, the Wiener filter methods, the Constrained least-squares filter method and the Lucy-Richardson algorithm.

Fig. 2 and Fig. 3 shows the corresponding *ISNR* and *MSE* value for restored images as a function of *l* for the proposed method and the mentioned classical methods. The figures illustrate that the quality of the restoration is as satisfactory as the classical methods or better from them (l<40 pixels). Realistically speaking, large motions do not occur frequently in radiography.

The results present in Fig. 4 – 6 refer to another original image.

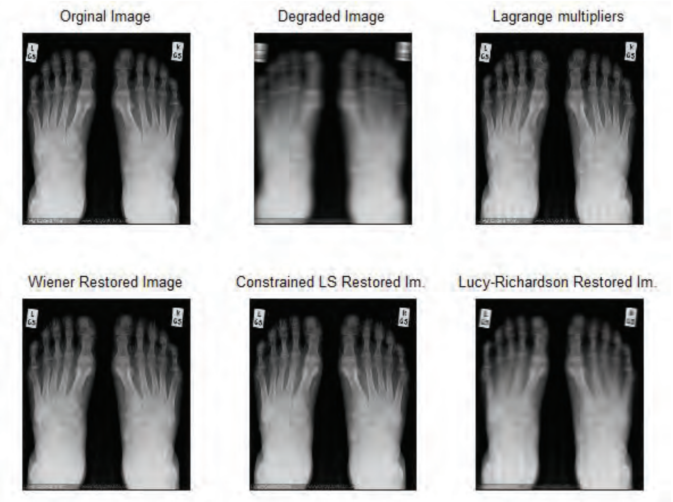

Fig. 4. Restoration in simulated degraded X-ray image for length of the blurring process, l=30

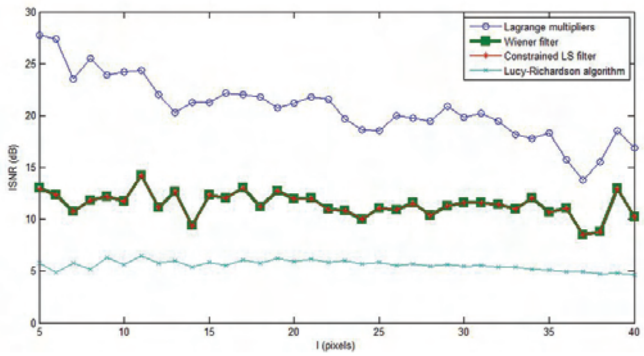

Fig. 5. Improvement in signal-to-noise-ratio vs. length of the blurring process in pixels

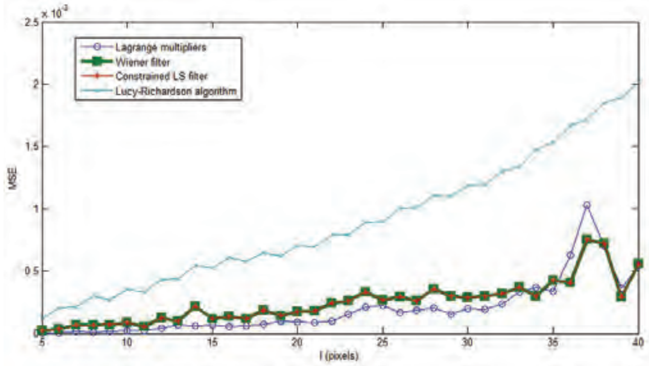

Fig. 6. Mean squared error vs. length of the blurring process in pixels

IV. Conclusion

We introduce a computational method, based on algorithm of Lagrange multipliers, to restore an X-ray that has been blurred by uniform linear motion.

We are motivated by the problem of restoring blurry images via well developed mathematical methods and techniques based on the Lagrange multipliers in order to obtain an approximation of the original image.

By using the proposed method, the resolution of the restored image remains at a very high level, although the main advantage of the method was found on the improvement *ISNR* that has been increased considerably compared to the other methods and techniques.

In this study, we present the results by comparing our method and that of the Wiener filter, Constrained least-squares filter and Lucy-Richardson algorithm, a well established methods used for fast recovered and high resolution restored images.

Obviously the proposed method is not restricted to restoration of X-ray images.

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