iCEST 2010 23 - 26 June 2010, Ohrid, Macedonia

Revisiting the Fractional-Order Hold Device

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Abstract – **This paper is concerned with the proper setting the fractional-order hold device for signal reconstruction in digital control systems. To demonstrate the features of the considered hold algorithm, using MATLAB®Simulink software both the time and frequency domain characteristics of the hold device are analyzed.**

Keywords **– Fractional-order hold device, signal reconstruction process, frequency response analysis, MATLAB®Simulink environment.**

I. Introduction

In computer-controlled systems, it is necessary to convert the control actions, calculated by the computer as a sequence of numbers, into a continuous-time signal that can be applied to the process. Theoretically, this kind of the reconstruction process can be done by employing an ideal low-pass filter. However, physically realizable data holds approximate in some sense an ideal low-pass filter. Higher-order hold circuits will generally reconstruct a signal more accurately than, in digital control systems commonly used, zero-order hold, but with some disadvantages related primarily to the possibility of its realization.

The paper is organized as follows. A brief review of the reconstruction process based on the fractional-order hold device is given in Section 2. Section 3 addresses to the quality analysis of the reconstruction process performed by using the frequency responses of the fractional-order hold device. Finally, in Section 4 some concluding remarks are given.

II. The Fractional-Order Hold

The sampling operation produces an amplitude-modulated pulse signal. The purpose of the hold operation is to reconstruct the original analog input signal. The hold circuit is designed to extrapolate the output signal between successive points according to some prescribed manner. So, the hold circuit that produces a staircase waveform is called a zero-order hold (ZOH). Recall that, because of its simplicity, the zero-order hold is commonly used in digital control systems. Higher-order hold circuits will generally reconstruct a signal more accurately than zero-order hold. For example, in the case of the first-order hold (FOH), the extrapolated function within a given interval is a straight line and its scope is determined by the values of the function at the sampling instants in the previous interval. The error generated in this way can be reduced by using only a fraction of the scope in the previous interval, as shown in Fig. 1b. This is achieved by using the fractional-order hold (FROH) device, given in Fig. 1a under the assumption that the value of α ranges from zero to unity [1].

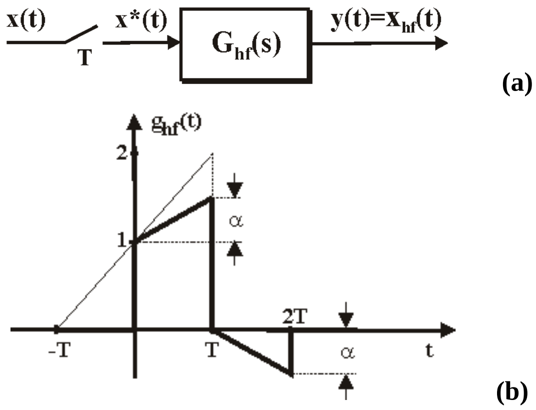

Fig. 1. Fractional-order hold device: (a) Block-diagram; (b) Impulse response.

The transfer function of the fractional-order hold can be shown to be [2]

$$G_{\mathrm{hf}}(s)=\left(1-\alpha\,\mathrm{e}^{-Ts}\right)G_{\mathrm{h0}}(s)+\frac{\alpha}{T}G_{\mathrm{h0}}^{2}(s)\,. \qquad (1)$$

After some simplifications, the above transfer function can be rewritten as follows

$$G_{\mathrm{hf}}(s)=\alpha G_{\mathrm{h1}}(s)+\left(1-\alpha\right)G_{\mathrm{h0}}(s)\ , \qquad (2)$$

where $G_{\mathrm{h0}}(s)$ and $G_{\mathrm{h1}}(s)$ are transfer functions of zero-order hold and first-order hold, respectively.

A. Discretizing the Models of the Analog Plants by using the Fractional-Order Hold

For simplicity, without loss of generality, consider the n th-order single-input single-output control object given by

$$\begin{aligned}\dot{\mathbf{x}}(t)&=\mathbf{A}_{\mathrm{c}}\mathbf{x}(t)+\mathbf{b}_{\mathrm{c}}u(t)\\ c(t)&=\mathbf{d}\mathbf{x}(t)\ ,\end{aligned} \qquad (3)$$

where $\mathbf{A}_{\mathrm{c}}\in\Re^{n\times n}$, $\mathbf{b}_{\mathrm{c}}\in\Re^{n\times 1}$, and $\mathbf{d}\in\Re^{1\times n}$.

It is convenient to introduce the realization sets as follows [3]

$$\mathrm{S}_{\mathrm{c}}\overset{\mathrm{def}}{=}\left\{\left(\mathbf{A}_{\mathrm{c}},\mathbf{b}_{\mathrm{c}},\mathbf{d}\right):G_{\mathrm{c}}(s)=\frac{N_{\mathrm{c}}(s)}{D_{\mathrm{c}}(s)}=\mathbf{d}\left(s\mathbf{I}-\mathbf{A}_{\mathrm{c}}\right)^{-1}\mathbf{b}_{\mathrm{c}}\right\}. \qquad (4)$$

Suppose a fractional-order hold is connected to the input of the object (4). Then $u(t)$ is given by

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$$u(t)=u(kT)+\alpha\frac{u(kT)-u\big[(k-1)T\big]}{T}(t-kT) \tag{5}$$

$$kT\le t<(k+1)T\,,\quad k=1,2,\ldots$$

where T denotes sampling period, and α is the adjustable gain of the hold device. The resulting sampled-data system can be described by the equations

$$\begin{aligned}\mathbf{x}\big[(k+1)T\big]&=\mathbf{A}_{\mathrm{q}}\mathbf{x}(kT)+\mathbf{b}_{\mathrm{q}\alpha}^{+}u(kT)+\mathbf{b}_{\mathrm{q}\alpha}^{-}u\big[(k-1)T\big]\\ c(kT)&=\mathbf{d}\mathbf{x}(kT)\ ,\end{aligned} \tag{6}$$

where

$$\begin{aligned}\mathbf{A}_{\mathrm{q}}&=\mathrm{e}^{\mathbf{A}_{\mathrm{c}}T}\\ \mathbf{b}_{\mathrm{q}\alpha}^{+}&=\int_0^T\Big(1+\alpha-\alpha\frac{t}{T}\Big)\mathrm{e}^{\mathbf{A}_{\mathrm{c}}\tau}\mathbf{b}_{\mathrm{c}}\,\mathrm{d}\tau\\ \mathbf{b}_{\mathrm{q}\alpha}^{-}&=\int_0^T\alpha\Big(\frac{t}{T}-1\Big)\mathrm{e}^{\mathbf{A}_{\mathrm{c}}t}\mathbf{b}_{\mathrm{c}}\,\mathrm{d}t\ .\end{aligned} \tag{7}$$

We denote the digital model (6) by $S_{\mathrm{q}\alpha}$, which can be rewritten in the form of the ordinary discrete-time state equation as

$$\begin{aligned}\begin{bmatrix}\mathbf{x}\big[(k+1)T\big]\\ u(kT)\end{bmatrix}&=\begin{bmatrix}\mathbf{A}_{\mathrm{q}} & \mathbf{b}_{\mathrm{q}\alpha}^{-}\\ \mathbf{0} & 0\end{bmatrix}\begin{bmatrix}\mathbf{x}(kT)\\ u\big[(k-1)T\big]\end{bmatrix}+\begin{bmatrix}\mathbf{b}_{\mathrm{q}\alpha}^{+}\\ 1\end{bmatrix}u(kT)\\ c(kT)&=\big[\mathbf{d}\quad 0\big]\begin{bmatrix}\mathbf{x}(kT)\\ u\big[(k-1)T\big]\end{bmatrix}\ .\end{aligned} \tag{8}$$

Thus,

$$S_{\mathrm{q}\alpha}\overset{\mathrm{def}}{=}\left\{\big(\mathbf{A}_{\mathrm{q}\alpha},\mathbf{b}_{\mathrm{q}\alpha},\mathbf{d}\big):G_{\mathrm{q}\alpha}(z)=\frac{N_{\mathrm{q}\alpha}(z)}{D_{\mathrm{q}\alpha}(z)}=\mathbf{d}\big(z\mathbf{I}-\mathbf{A}_{\mathrm{q}\alpha}\big)^{-1}\mathbf{b}_{\mathrm{q}\alpha}\right\} \tag{9}$$

where

$$G_{\mathrm{q}\alpha}(z)=\mathcal{Z}\,\mathcal{L}^{-1}\left\{G_{\mathrm{hf}}(s)\cdot G_{\mathrm{c}}(s)\right\}\ , \tag{10}$$

represents FROH equivalent model for the continuous-time system (4) at sampling interval T. However, it is interesting to note that the model can be converted from continuous-time to discrete-time using a matrix exponential as follows [4], [5]:

$$\left[\begin{array}{c:c:c}\mathbf{A}_{\mathrm{q}} & \mathbf{b}_{\mathrm{q}0} & -\mathbf{b}_{\mathrm{q}1}\\ \hdashline \mathbf{0} & 1 & 1\\ \hdashline \mathbf{0} & 0 & 1\end{array}\right]=\exp\left[\begin{array}{c:c:c}\mathbf{A}_{\mathrm{c}}^{\mathrm{T}} & \mathbf{b}_{\mathrm{c}}^{\mathrm{T}} & 0\\ \hdashline \mathbf{0} & 0 & 1\\ \hdashline \mathbf{0} & 0 & 1\end{array}\right]\ , \tag{11}$$

where

$$\begin{aligned}\mathbf{b}_{\mathrm{q}\alpha}^{-}&=\alpha\,\mathbf{b}_{\mathrm{q}1}\\ \mathbf{b}_{\mathrm{q}\alpha}^{+}&=\mathbf{b}_{\mathrm{q}0}-\alpha\,\mathbf{b}_{\mathrm{q}1}\ .\end{aligned} \tag{12}$$

The same discrete transfer function $G_{\mathrm{q}\alpha}(z)$, as that given by (10), can be also derived from (6)-(8) and represented in form of

$$G_{\mathrm{q}\alpha}(z)=\frac{N_{\mathrm{q}\alpha}(z)}{D_{\mathrm{q}\alpha}(z)}=\frac{(z-1)\det\begin{bmatrix}z\mathbf{I}-\mathbf{A}_{\mathrm{q}} & -\mathbf{b}_{\mathrm{q}\alpha}^{+}\\ \mathbf{d} & 0\end{bmatrix}+z\det\begin{bmatrix}z\mathbf{I}-\mathbf{A}_{\mathrm{q}} & -\mathbf{b}_{\mathrm{q}0}\\ \mathbf{d} & 0\end{bmatrix}}{z\det\big[z\mathbf{I}-\mathbf{A}_{\mathrm{q}}\big]} \tag{13}$$

It is well-known that the zeros of a discrete-time system can be classified into two categories: those which correspond to the zeros of continuous-time system, and so called discretization zeros introduced by the chosen discretization method. As it can be found in the literature, the significant improvements in the transient performance of the closed-loop system can be obtained by using a properly adjusted FROH, instead ZOH or FOH devices. This could be achieved by using the negative values of α, because it results in a more stable discretization zeros [6].

B. One Realization of FROH Device

In general, only zero-order hold devices are implemented in hardware [7]. Higher-order hold algorithms are implemented in computer software since it is most difficult physically to implement higher-order hold devices, as fractional-order hold for example. Also, the realizations of some approximated versions of FROH can be found in the literature. It is suitable to emphasize that the lack a number of real experiments with the special hold devices is mainly due to the difficulties encountered in its implementation with the processing speed and configurability that real-time digital controllers require [8].

As a block diagram, an analog FROH device is presented in Fig. 2. This is a straight implementation of Eq. (2). The proposed realization is based on two sample and zero-order hold devices and is suitable for both the computer simulation and real experimentation.

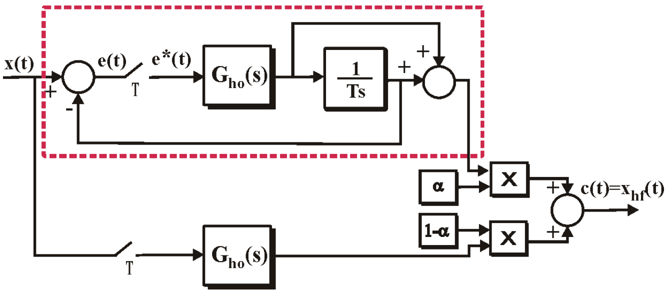


Fig. 2. Block-diagram of the fractional-order hold device

C. Rational approximation of $G_{\mathrm{hf}}(s)$

Since the transfer function (1) is not a rational function, it is suitable to find its finite-dimensional rational approximation. Among the many methods, PADÉ approximations are widely used to approximate by a rational function the dead-time given in continuous-time control systems with transfer function $e^{-s\tau}$. Instead of the standard approximation, the PADÉ approximation, where the numerator degree is one less than the denominator degree, is recommended. Recall that the PADÉ approximation of order (m,n) is defined to be a rational function $R_{m,n}(s)$ expressed in a fractional form [9]:

$$e^{-\tau s} \approx R_{m,n}(\tau s) = \frac{N_m(\tau s)}{D_n(\tau s)} \qquad (14)$$

where

$$N_m(\tau s) = \sum_{k=0}^{m} (-1)^k \frac{(m+n-k)!m!}{(m+n)!k!(m-k)!} \tau^k s^k \qquad (15)$$

$$D_n(\tau s) = \sum_{k=0}^{n} \frac{(m+n-k)!n!}{(m+n)!k!(n-k)!} \tau^k s^k . \qquad (16)$$

Note that the high-order PADÉ approximations produce transfer functions with clustered poles. Because such pole configurations tend to be very sensitive to perturbations, PADÉ approximations with order $n > 10$ should be avoided.

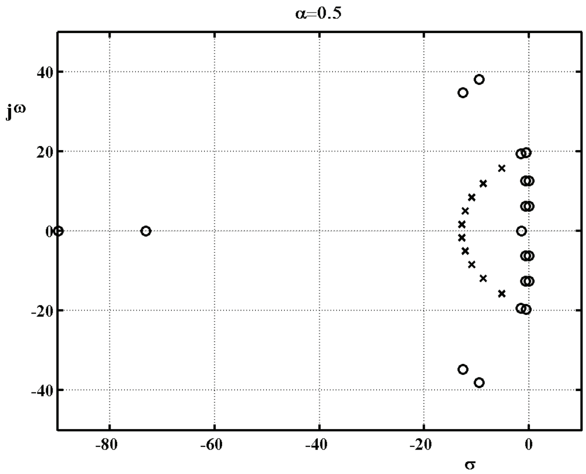


Fig. 3. Pole-zero configuration of the $G_{f\,9,10}(s)$ approximation of transfer function $G_{hf}(s)$ for $\alpha = 0.5$

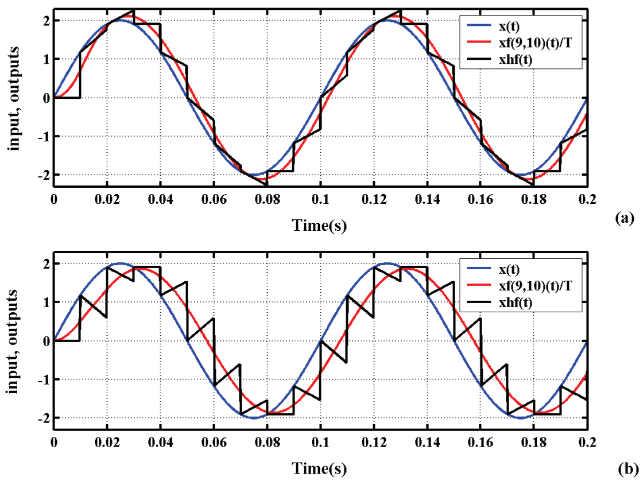


Fig. 4. Responses of FROH to sine wave input for different values of parameter α : (a) $\alpha = 0.5$; (b) $\alpha = -0.5$.

In this paper, the approximation $R_{9,10}(Ts)$ for the transfer function e^{-Ts} is adopted. Fig. 3 shows the pole-zero configuration of the $(9,10)$ – order PADÉ approximation for the transfer function $G_{hf}(s)$ and $\alpha = 0.5$ in (1). Traces in Figs. 4 correspond to two different values of parameter α, and visualize the outputs $x_{hf}(t)$ of the sampler and fractional-order hold given in Fig. 2, as well as the responses of the rational function $G_{f\,9,10}(s)$, corrected by T according to sampling theory. Note that the input $x(t)$ is a sinusoidal signal, as well as the sampling frequency is relatively low of 10 samples per cycle. It can be seen that in the case of negative value of parameter α the fractional-order hold introduced a much larger time lag to the input signal.

III. FREQUENCY RESPONSES OF FRACTIONAL-ORDER HOLD DEVICE

The frequency response data can be visualized in two different ways: via the NYQUIST diagram or via the BODE plots. Note that the BODE diagrams of the hold devices can be found mostly in the usual textbooks in the field of digital control systems.

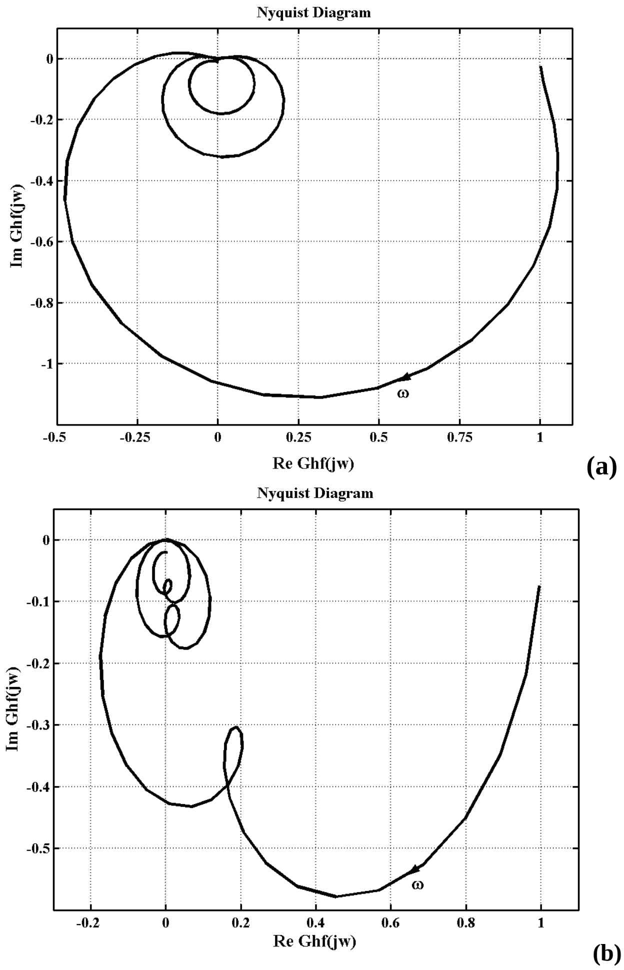
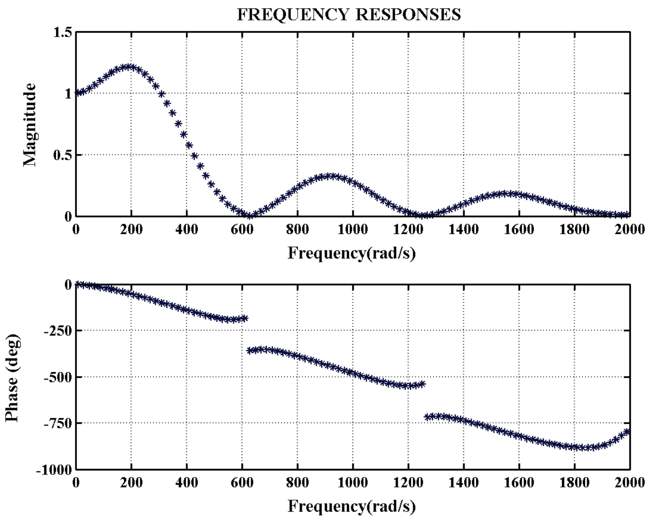
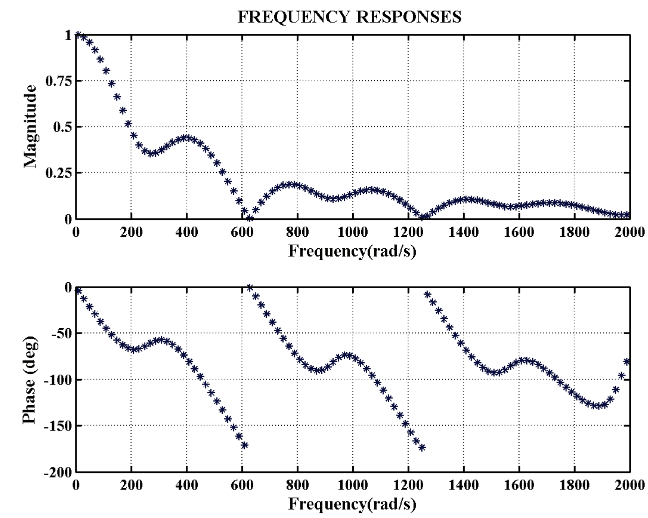


Fig. 5. The NYQUIST plots of the fractional order hold device for different values of parameter α : (a) $\alpha = 0.5$; (b) $\alpha = -0.5$.

In Figs. 5a and 5b it is possible to recognize the typical amplitude-phase frequency (NYQUIST) plots for systems of infinite dimension. Figs. 6a and 6b show plots of the magnitude and phase characteristics of the fractional-order hold device for the sampling period $T = 0.01\,\mathrm{s}$ and two values of the adjustable parameter $\alpha = \pm 0.5$. It can be seen that the fractional-order hold for both values of α does not have the ideal low-pass filter characteristics. Namely, the FROH device allows significant transmission above NYQUIST frequency $\omega_s/2 = \pi/T$. Thus, the signal must be low-pass-filtered before the sampling operation, so that the frequency components above the NYQUIST frequency are negligible. Moreover, in the case of negative value of parameter α, both amplitude and phase characteristics are significantly distored, even for frequencies which belong to the NYQUIST area of frequencies.



(a)



(b)

Fig. 6. Frequency responses of fractional order hold device for different values of parameter α : (a) $\alpha = 0.5$; (b) $\alpha = -0.5$.

Note that the time and frequency domain characteristics given in Figs. 4 and 6 for different values of parameter α fully correspond to each other.

IV. CONCLUSION

The reconstruction using the fractional-order hold is not very common in practice because its dynamics is relatively complicated to implement. Both the time and frequency domain characteristics of the considered hold device are analyzed for different values of the adjustable parameter. To generate the frequency response plots the well-known MATLAB® procedures are used. Bearing in mind that these procedures have several limitations specially in the case of nonrational transfer function, it would be suitable to consider the possibility of the development of some new procedures to generate the BODE and NYQUIST frequency response plots for nonrational transfer functions.

ACKNOWLEDGEMENT

This work was supported in part by Ministry of Science and Technological Development of the Republic of Serbia under the Grant ET 11029/2008-2010.

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