FIR Filter Design using Compressed Cosine Polynomial Approximation

Peter Apostolov[1]

Abstract – **This paper considers a new method for FIR filters design. The method uses an L_∞ optimality norm. To achieve a better approximating effect, a new modulating function which compresses the oscillations of the cosine is proposed. A parameter sets the gradient of the modulating function, with respect to the oscillations' compression. The approximating polynomial is carried out using Remez' exchange algorithm. An optimal polynomial with lowest possible (four) degree, that approximates an ideal filter's response with high precision is proposed. With the proposed method an FIR filter with arbitrary specifications can be designed. Design example of low pass FIR filter with a minimization of calculation is performed. The obtained filter's response is close to the ideal low pass filter response.**

Keywords – **FIR digital filters, Frequency response, Polynomial approximation.**

I. Introduction

The filters' design is a mathematical problem for an approximation of the ideal filter's response. This is an ideal function, comprising rectangular shape with two areas: pass band and stop band. The analytical expression of the ideal low pass filter response is.

$$D(\omega)=\begin{cases}1, \omega\in[0,\omega_c]\\0, \omega\in(\omega_c,\pi]\end{cases}, \quad (1)$$

where ω_c is the cut-off frequency. In the FIR filter design, the approximation function is a cosine polynomial

$$A(\omega)=\frac{1}{\sqrt{2}}a_0+\sum_{n=1}^{m}a_n\cos(n\omega)\triangleq\sum_{n=0}^{m}a_n\cos(n\omega). \quad (2)$$

The coefficients of the filter's impulse response are determined by the coefficients of the polynomial. The difference between the ideal function and the approximating polynomial defines the error function.

$$E(\omega)=D(\omega)-A(\omega),\ \omega\in[0,\pi]. \quad (3)$$

When the approximation is optimal, the graph of the approximating polynomial oscillates with equal amplitude close to the ideal function's graph. The amplitude of the oscillations determines the approximation's error. The three most popular norms used in FIR design are as follows:

-- weighted L_1 norm. The approximation's error is

$$\|E(\omega)\|_1=\int_0^\pi W(\omega)|E(\omega)|\,d\omega; \quad (4)$$

-- L_2 error - weighted integral least-squares norm

$$\|E(\omega)\|_2^2=\int_0^\pi W(\omega)|E(\omega)|^2\,d\omega; \quad (5)$$

-- L_∞ error - weighted Chebyshev's norm

$$\|E(\omega)\|_\infty=\max_{\omega\in[0,\pi]}\left[W(\omega)|E(\omega)|\right]. \quad (6)$$

In the all above cases $W(\omega)$ is a positive weighting function, used in order to weight certain frequencies. When $W(\omega)=1$, the maximal error in the pass band and stop band is equal. In this case the approximation using L_∞ norm is equiripple.

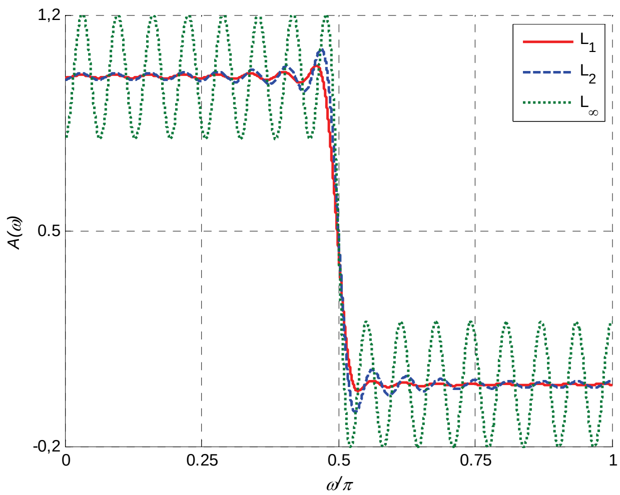

Fig. 1. Approximations with 32-degree polynomials using L_1, L_2 and L_∞ norms

Figure 1 shows a comparison of 32-degree approximating polynomials using L_1, L_2 and L_∞ norms. It is seen that in the FIR filters design a suitable trade-off between flatness and the transition bandwidth must be made. In all the criteria, the functions have the oscillations in the pass band and stop band. These oscillations are undesirable. The goal in the design is to

[1]Peter Stoyanov Apostolov is from the Institute for Special Technical Equipment - MI, Bulgaria, Sofia, 1799, POB 83; e-mail: p_apostolov@abv.bg.

obtain a rectangular shape of the ideal filter's response, that is maximally flat pass band and stop band, and narrowest possible transition band.

In L_1 and L_2 cases the oscillations increase near to the function's transition band. This is due to the Gibbs' phenomenon [1]. The filter design using L_2 criterion is studied in details in many publications. A various methods for reduction of the Gibbs' phenomenon have been proposed: with window functions, "don't care" transition band, optimal change the transition band etc.

Methods for L_2 FIR filters design, where the function's overshot near the transition band is constrained, are presented in [2]. The method success is in reducing the Gibbs' phenomenon. The design is known as "Constrained Least Square" filters. The synthesis is performed with iterative algorithm, having set the maximum value of the overshot. As a result an alignment of the oscillations' amplitude near the transition band is obtained. This leads to the transition's band expansion. The transition band is either not set as an input parameter in the synthesis, or set only at one of the frequencies that define it. The transition bandwidth is the result of the approximation, and is called "induced" transition band. In [3] computationally efficient method for L_2 FIR filter design is proposed.

With approximation using L_1 norm [4] filters with flatter response, but with wider transition band are obtained (Fig.1).

The L_∞ norm is the most suitable for the filter design problem. The design is performed with the well known Parks-McClellan's method [5]. This is minimax approximation using the Chebyshev's norm. With this method, the transition band can be accurately defined and arbitrarily narrow. However this increases the approximation's error and the amplitude of oscillations (Fig.1). It was found that with the same specification (pass band ripple, stop band attenuation and transition band width) with Parks-McClellan's method, an approximation with polynomial of lowest degree has been done. That means that the L_∞ filters will be implemented with the least number of coefficients.

A new polynomial approximation using L_∞ norm producing FIR filters with response close to the ideal will be proposed in the article.

II. Background

As noted in all criteria, the approximating polynomial is a sum of cosines. Approximation using L_∞ norm can be done if the polynomial is a linear combination of Chebyshev's polynomials (Fig.2)

$$A(\omega) = \sum_{n=0}^{m} a_n \cos\left(n \arccos\frac{\omega}{\pi}\right), \ \omega \in [0, \pi]. \qquad (7)$$

The approximation is optimal, but inefficient, because the function's graph compresses its oscillations on both ends of the definition domain, while in the transition band they are most sparse. Thus, a high slope of the function in the transition band can not be obtained.

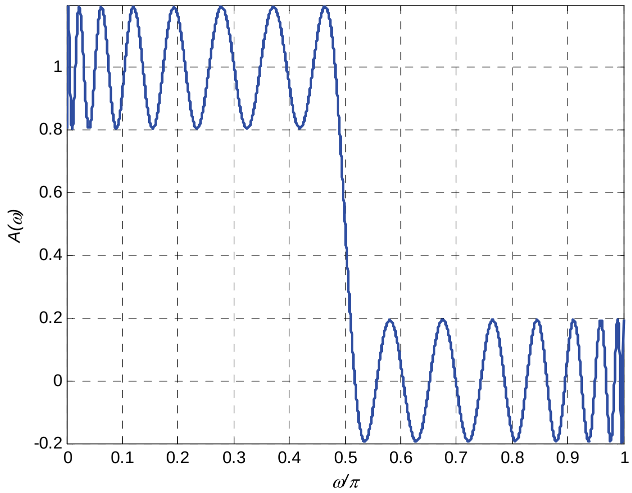

Fig. 2. Approximation with linear combination of Chebyshev's polynomials

From (7) it is obvious that the oscillation's compression is owing to the $\arccos(.)$ function, which modulates the $\cos(.)$ function's argument, as is shown in figure 3.

It is logical to assume if another modulating function with inverse slope to $\arccos(.)$, and higher gradient of the graph is applied, a polynomial with higher oscillations' compression in the transition band, and greater slope will be obtained. A new function called "Third basis function" in [6] is proposed.

$$y(x) = \cos\left\{\frac{m\pi}{2}\left[\tanh\left(\beta x - \frac{\beta}{2}\right) + 1\right]\right\}; x \in [0,1]. \qquad (8)$$

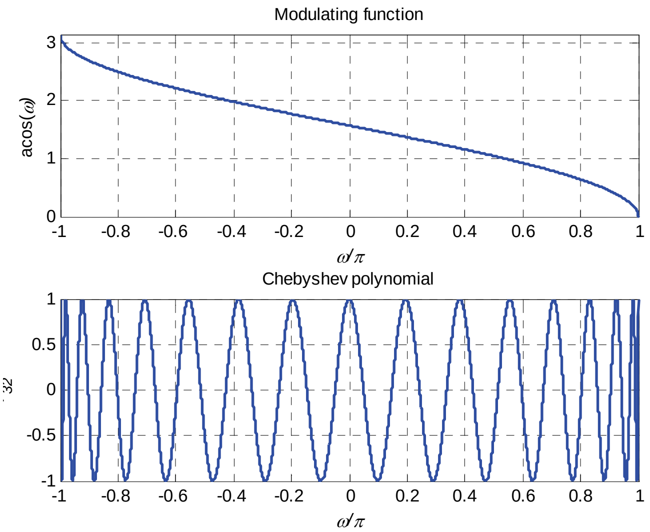

Fig. 3. Modulating function $\arccos(.)$ and Chebyshev's polynomial

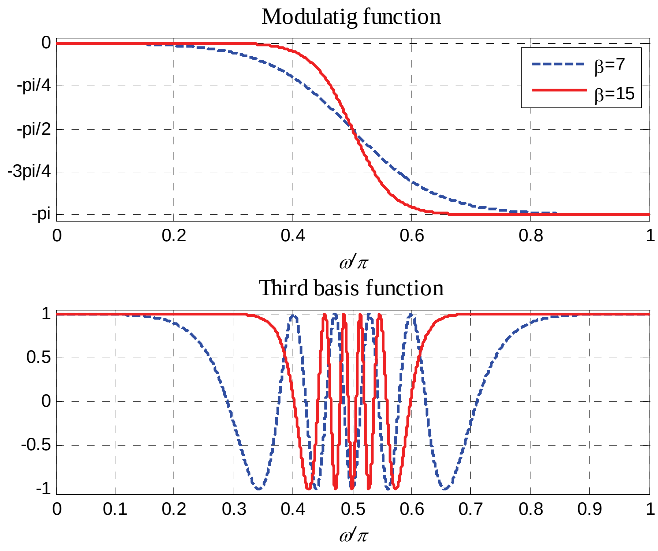

Fig. 4. Modulating function $\tanh(.)$ and Third basis function, $m = 10$

The modulating function is $\tanh(.)$, which comprises parameter $\beta > 4$. Changing the parameter sets the slope of the modulating function, and the oscillations' compression, as is shown in figure 4. Similar to the Parks-McClellan's method, the approximating polynomial with Remez' exchange algorithm [7] is performed. Concerning the specific requirements of the algorithm, the degree of the polynomial *m* is an even number. For the purposes of the FIR filters design the polynomial has the form.

$$A_m(\omega) = \sum_{k=1}^{m+1} a_k \cos\left\{(k-1)\frac{\pi}{2}\left[\tanh\frac{2\beta(\omega-\omega_c)}{\omega_s}+1\right]\right\}, \quad (9)$$

where $\omega \in [0, \omega_s/2]$, and ω_s is the sampling frequency. The goal in the proposed method is that the amplitude of the function's overshot decreases when the parameter β increases, *without changing the width of the transition band, or increasing the degree of the polynomial.*

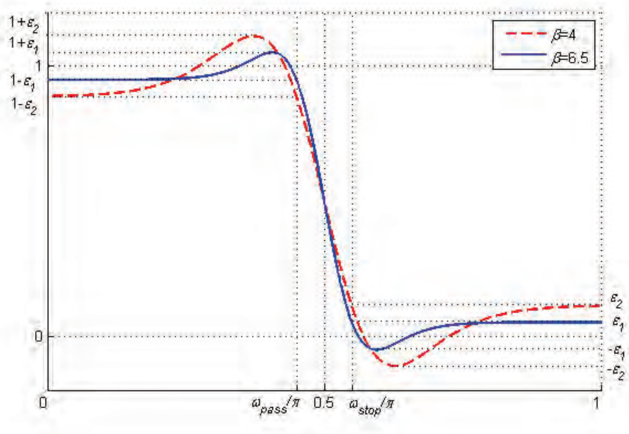

Fig.5. Approximation's error ε depending of the parameter β by fixed transition band

In the other methods it is not so. Figure 5 shows the graph of polynomial with the lowest possible (four) degree for two values of the parameter β. The pass band ripple *DP* and stop band attenuation *DS* are determined by the approximation's error ε as follows:

$$DP = 20\lg(1-|\varepsilon|)\ \text{dB};\ DS = 10\lg|\varepsilon|\ \text{dB}. \quad (10)$$

III. Design Example

The properties of the filters are illustrated through low pass FIR filter example design.

Filter's specification: Cut-off frequency $\omega_c = 0.2\pi$ rad/s; transition band width $\Delta\omega_c = 0.005\pi$ rad/s, sampling frequency $\omega_s = 2\pi$ rad/s; stop band attenuation $DS \geq 60$ dB, power of the polynomial $m = 4$; weighting function $W = 1$.

Parameter's value β can be determined approximately

$$\beta = \frac{\omega_s DS}{32.9\Delta\omega_c} \approx 730. \quad (11)$$

The polynomial's coefficients are determined with the Remez' algorithm: $a_1 = 0.5$; $a_2 = 0.5628$; $a_3 = a_5 = 0$; $a_4 = -0.0628$; and approximation's error $\varepsilon = 7.82\text{e-}7$. The exact values of the pass band ripple and stop band attenuation are calculated by (10); $DP = -6.8\text{e-}6$ dB; $DS = -61.07$ dB.

Equation (9) is the filter's magnitude response. It should be noted that the coefficients of the impulse response are not obtained directly from polynomial's coefficients as by the other methods, as the argument of the function $\cos(.)$ contains other, non-linear function. The impulse response can be determined with IFFT of the 2^N samples of the magnitude response. The filter's design is done using a method, known as "frequency sampling filter" [8]. It is based on FFT in 2^N samples, where N is an integer positive number. For the purposes of the design a window function with 2^{N-1} values of the magnitude response (9) are calculated. The signal filtration is performed with convolution between FFT of the input signal and the window function. The filter's structure is shown in figure 6. It should be noted that a larger number of samples should be determined to realize a narrow transition band. In this particular case the appropriate value is $N = 13$. Therefore the window function consists of 4096 values, which determine the filter's length. Figure 7 shows the filter's magnitude response.

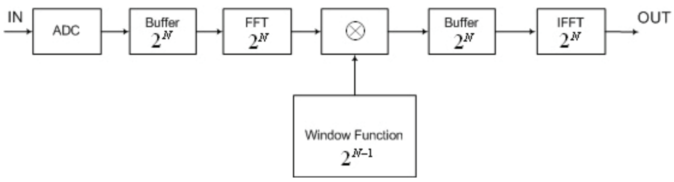

Fig.6. Filter's structure

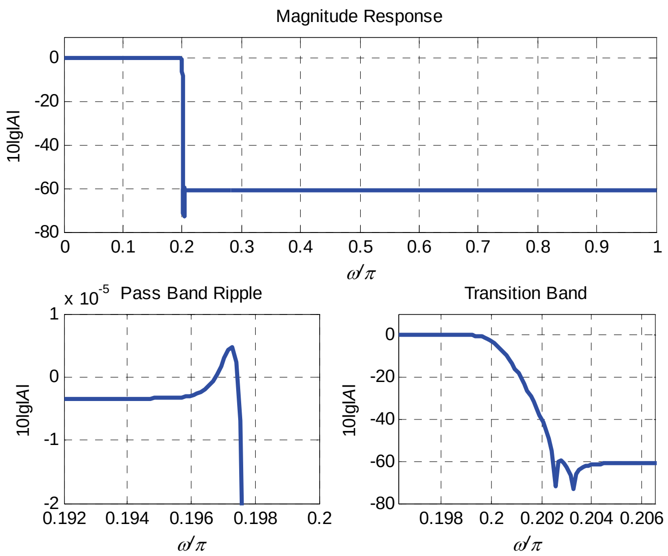

Fig.7. Low pass FIR filter magnitude response

It is obvious that the characteristic is very close to the rectangular shape of the ideal low pass response. The magnitude response is maximally flat. It is seen that large parts of pass band and stop band are constant: $1-\varepsilon$ and ε. Since ε is very small number (approximately 1e-6), it can be assumed that most of the values of the magnitude response in the pass band are equal to one, and in the stop band they are zeros. This circumstance leads to significant reduction of the calculation process of the filter. For the performance of the signal's filtration, it is necessary to execute convolution only in the frequency band slightly wider than the transition band. Of course, it must include the values of the actual transition band. The rest of the pass band frequencies are transferred directly to output buffer (since it is not necessary to multiply by one), and those of the stop band are equal to zero. Figure 8 shows the magnitude response of the same filter with described calculations' minimization. Thus the filter of the considered example is realized with only 32 multiplications.

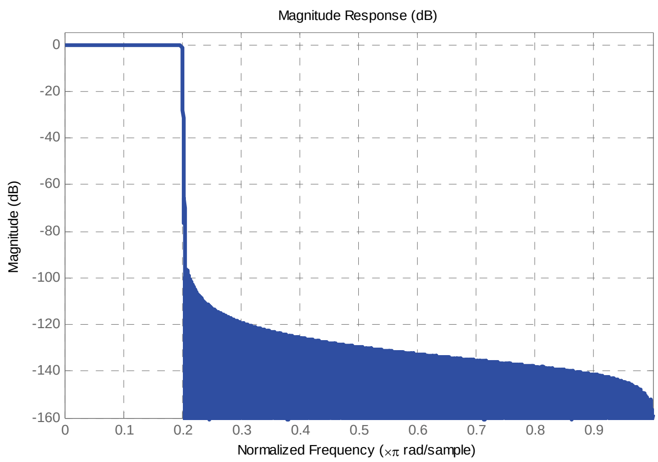

Fig.8. Magnitude response with minimization of the calculations

IV. Conclusion

FIR filters with responses close to ideal filter's response can be designed with the proposed method. A basis function that compresses the oscillations of the function in the transition band is proposed. The method gets its name from this compression. The compression ratio is determined by the parameter β. The approximation is carried out using L_∞ norm that most closely approximates the transition band. The approximating polynomial is obtained by Remez' algorithm, which has fast convergence and low computational complexity. In this case, the low degree of the polynomial (four) involves iterative solving of a system of six linear equations. In other methods they are much more. The method has very low computational cost and design complexity, which is an important advantage. With 4th degree polynomial FIR filter with arbitrary specifications in a *fixed transition band* can be designed. The approximation error ε depends on the parameter β, and defines the pass band ripple and the stop band attenuation. No other polynomial of 4th or lower degree with better approximating properties is known till date. When the width of the transition band is equal to zero and parameter $\beta=\infty$, the polynomial's graph coincides with the ideal response's rectangular shape. Of course, this is impossible in reality. The theoretical design possibilities are limited by the computer calculation accuracy. The implementation of the proposed method makes sense in the design of filters with extreme, close to ideal response. The repeated reduction of the computational operations (Fig.8) is effective in approximations with values of ε close to zero and a very narrow transition band. The proposed method may be a good alternative in several applications in the FIR filters' design.

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