Displacement Signal Error Approximation for Uncorrelated Noise of Laser Illuminated Object

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Abstract – **The displacement signal error for uncorrelated noise of laser illuminated object (target) is analyzed. A mathematical relation for probability density function of displacement signal for a laser illuminated object is given. The displacement signal error of laser illuminated object versus signal-to-noise ratio (SNR) and the mean value displacement signal ($\bar{\varepsilon}$) is derived. The displacement signal error is derived in closed form for given value of the spot position probability 0.5, using a system for symbolic computing Mathematica.**

Keywords – **Quadrant photodiode, Displacement signal, Probability density function, Displacement signal error.**

I. INTRODUCTION

In industry and army applications in which accurate positioning of the objects are very important [1]. The main characteristic of those applications is accuracy of positioning measurement of an object. Laser positioning systems have a special treatment, because their resolution is better than radar and other positioning systems. A number of lasers positioning applications include tracking of laser illuminated target and measurement of its angular position [2], [3], estimation of satellite vibration effect [4], and measurement of multidimensional displacement in space [5]. These and other applications can use position sensitive detectors with quadrant photodiode (QPD) [6] or lateral effect photodiode (LEP) [7] to measure lateral displacement in two perpendicular planes. Theoretical displacement signal analysis had been discussed in [1], [2], [3], [8], and [9]. Those papers have presented that the main parameter is signal-to-noise ratio, which limits positioning accuracy in laser systems with quadrant photodiode. Also, they have found that the position error is changed with mean value of displacement signal [8]. The expression for probability density function of displacement signal for Gaussian noise distribution is presented in [9].

In this paper a new approach to estimate the position error of laser illuminated object by quadrant photodiode is given. This is statistically based approach, where the displacement signal error is obtained from probability density function of displacement signal.

II. PROBABILITY DENSITY FUNCTION OF DISPLACEMENT SIGNAL

Consider a light spot, of radius r, falling on QPD with radius a as shown in Figure 1. The position of the photodiode center is at (0, 0) and the center of the spot at (x, y), where x, and y are displacement positions of the spot center along the X and Y axes, respectively.

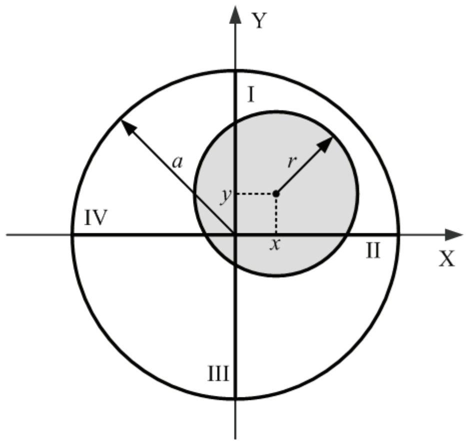

Fig. 1. Geometry of the light spot position on the QPD surface

Displacement signals depend on the spot positions in both directions. These signals can be obtained by processing the output currents of four photodiode quadrants. The normalized displacement signal ε in the X or Y direction is the function of currents of the quadrants proportional to the area of the spot falling on the quadrant [10].

We assume that each quadrant generate noise with Gaussian distribution with zero mean value. Then the probability density function (*pdf*) of displacement signal is obtained, based on known theory for jointly normal random variables, probability density functions and some transformations [11].

Useful way to analyze *pdf* of displacement signal $f(\varepsilon)$ is substituting the mean value pair with mean value of the displacement signal ($\bar{\varepsilon}$) and the total signal-to-noise ratio in the channel is *SNR*. For uncorrelated noise, $f(\varepsilon)$ is obtained in [10],

$$f(\varepsilon) = \frac{1}{\pi(1+\varepsilon^2)} \cdot \exp\left(-\frac{SNR}{2}(1+\bar{\varepsilon}^2)\right) \times \left[1+\sqrt{\frac{\pi}{2}} \cdot B\left(erf\left(\frac{B}{\sqrt{2}}\right)\right)\exp\left(\frac{B^2}{2}\right)\right] \quad (1)$$

where the parameter B is

$$B = \frac{1+\varepsilon\bar{\varepsilon}}{\sqrt{1+\varepsilon^2}}\sqrt{SNR} \quad (2)$$

Probability density function $f(\varepsilon)$ is the function of the mean value $\bar{\varepsilon}$, the displacement signal ε and signal-to-noise ratio

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SNR, as shown (1) and (2). Diagrams of *pdf* from (1) for SNR=10 dB, and three values of $\bar{\varepsilon}$ are given in Figure 2 as a function of ε.

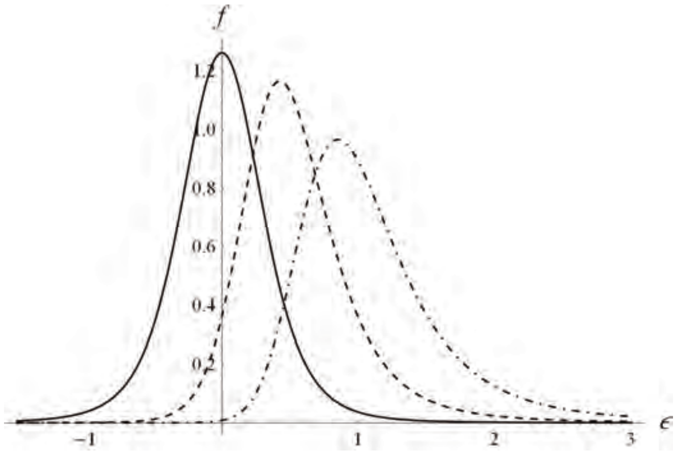

Fig. 2. Probability density function of displacement signal for *SNR*=10 dB and $\bar{\varepsilon}$ =0 (solid line), $\bar{\varepsilon}$ =0.5 (dashed line), $\bar{\varepsilon}$ =1 (dot-dashed line).

Figure 2 shows that the *pdf* maximum decreases and the *pdf* width increases by increasing $\bar{\varepsilon}$. Maximum width and minimum amplitude of *pdf* was obtained for $\bar{\varepsilon}$ =1. From Figure 2, it can be seen that the probability density function of the displacement signal $f(\varepsilon)$ for uncorrelated noise does not have the maximum value at the mean value of the displacement signal (see dot-dashed line for $\bar{\varepsilon}$ =1).

III. The Spot Position Probability

The displacement signal error presents a small range $\Delta\varepsilon$ around the mean value $\bar{\varepsilon}$ in which the spot position probability is calculated. We define the spot position probability *Pp*

$$P_p = \int_{\bar{\varepsilon}-\Delta\varepsilon}^{\bar{\varepsilon}+\Delta\varepsilon} f(\varepsilon)d\varepsilon \qquad (3)$$

where $\Delta\varepsilon$ is the displacement signal error.

From (1), and (2), the spot position probability for the displacement signal ε in range $\Delta\varepsilon$ around $\bar{\varepsilon}$ becomes

$$P_p = \frac{Pp1 + Pp2}{2}$$
$$Pp1 = erf\left(\frac{\sqrt{SNR}}{\sqrt{2}}\frac{\Delta\varepsilon}{\sqrt{1+(\bar{\varepsilon}+\Delta\varepsilon)^2}}\right) \qquad (4)$$
$$Pp2 = erf\left(\frac{\sqrt{SNR}}{\sqrt{2}}\frac{\Delta\varepsilon}{\sqrt{1+(\bar{\varepsilon}-\Delta\varepsilon)^2}}\right)$$

Diagrams of *Pp* (4) are given in Figure 3 as a function of $\Delta\varepsilon$.

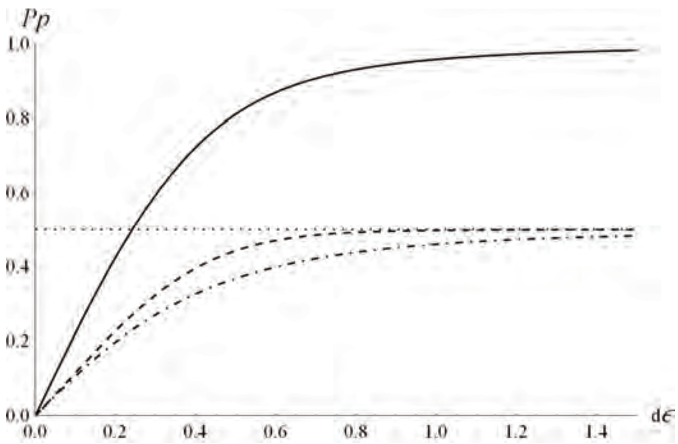

Fig. 3. Spot position probability for *SNR*=10 dB and $\bar{\varepsilon}$ =0.5 (solid line), *Pp*1 (dashed line), and *Pp*2 (dot-dashed line).

The spot position probability is the sum of Pp_1 and Pp_2 and it is not possible to find exact value of $\Delta\varepsilon$ in closed form for known *Pp*. An approximation of *Pp* in closed form is derived in [10]. In order to find another good approximation of $\Delta\varepsilon$ for specified value of *Pp*, we will analyze several examples. For the analysis purpose we are using a system for symbolic computing Mathematica [12].

IV. Displacement Signal Error

In order to determine displacement signal error for specified value of *Pp*, we enter the knowledge into Mathematica

$$\texttt{f[}\epsilon\texttt{\_, m\_ , SNR\_] :=} \frac{e^{-\frac{SNR\,(-m+\epsilon)^2}{2\,(1+\epsilon^2)}}\sqrt{SNR}\;(1+m\,\epsilon)}{\sqrt{2\pi}\;\left(1+\epsilon^2\right)^{3/2}}$$

$$\texttt{Pp[d}\epsilon\texttt{\_, m\_, SNR\_] :=} \frac{1}{2}\,\texttt{Erf}\left[\frac{d\epsilon\sqrt{SNR}}{\sqrt{2}\sqrt{1+(-d\epsilon+m)^2}}\right] + \frac{1}{2}\,\texttt{Erf}\left[\frac{d\epsilon\sqrt{SNR}}{\sqrt{2}\sqrt{1+(d\epsilon+m)^2}}\right]$$

where **m** is used for $\bar{\varepsilon}$, and **d**$\boldsymbol{\epsilon}$ for the error displacement $\Delta\varepsilon$. Next, we compute the value of $\Delta\varepsilon$ for specified value of *Pp*, say **c1**=0.5, using symbol **d** for $\Delta\varepsilon$. The squared error of the displacement signal is a linear function of the squared mean value of displacement, as it is shown in Figure 4.

```
SNR1 = 10;
m1 = 0;
c1 = 0.5;
t2 = {{m1^2, d^2} /. FindRoot[Pp[d, m1, SNR1] - c1 == 0, {d, 0.3}]}
Do[
  t1 = d^2 /. FindRoot[Pp[d, m1, SNR1] - c1 == 0, {d, 0.3}];
  t2 = Append[t2, {m1^2, t1}]
  , {m1, 0 + 0.1, 1, 0.1}]
```

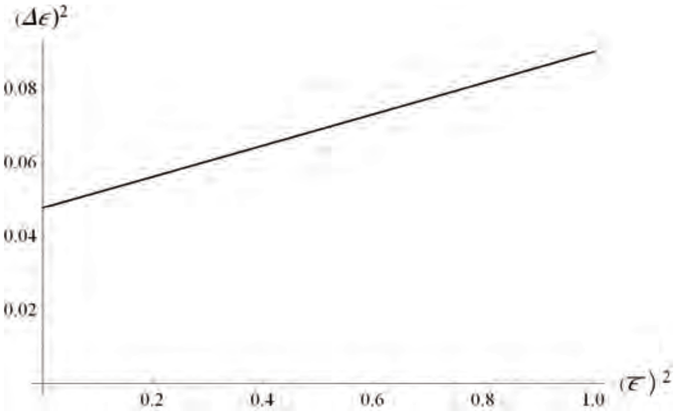

Fig. 4. The squared error of the displacement signal for *SNR*=10 dB is liner function of $\bar{\varepsilon}$.

The similar diagram is computed for different values of SNR

```
s[SNR_] := 10^(SNR/10)
SNR10 = 10.;
m1 = 0.5;
c1 = 0.5;
t2 = {{1/s[SNR10], d^2} /. FindRoot[Pp[d, m1, s[SNR10]] - c1 == 0, {d, 0.3}]}
Do[
 t1 = d^2 /. FindRoot[Pp[d, m1, s[SNR1]] - c1 == 0, {d, 0.3}];
 t2 = Append[t2, {1/s[SNR1], t1}]
 , {SNR1, SNR10 + 1., 30, 1}]
```

Figure 5 illustrates that the squared error of the displacement signal is a linear function of 1/*SNR*. Those properties of the displacement signal error, for specified value of *Pp*, gave us an idea to find a good approximation of $\Delta\varepsilon$ in terms of $\bar{\varepsilon}$ and *SNR*.

We define a new function **d2** as a linear function of $(\bar{\varepsilon})^2$ and **n**=1/*SNR*

```
d2 = (k0 + k1 m^2) n
```

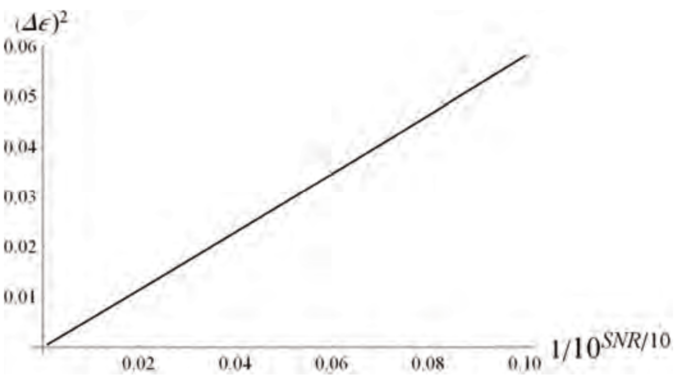

Fig. 5. The squared error of the displacement signal for $\bar{\varepsilon}$ =0.5 is liner function of $1/10^{SNR/10}$, *SNR* in dB.

Next, we compute the values for two characteristic cases $\bar{\varepsilon}$ =0 and $\bar{\varepsilon}$ =1, and for **n**=(1/*SNR*)=1/10.

```
d2m0 = d2 /. {m → 0, n → 1 / 10}
d2m1 = d2 /. {m → 1, n → 1 / 10}
```

```
k0/10
(k0 + k1)/10
```

Solving the equations, we determine two coefficients

```
d0 = d^2 /. FindRoot[Pp[d, 0, s[10]] - 0.5 == 0, {d, 0.3}]
0.047662
d1 = d^2 /. FindRoot[Pp[d, 1, s[10]] - 0.5 == 0, {d, 0.3}]
0.0900052
sol1 = Solve[{d2m0 == d0, d2m1 == d1}, {k0, k1}]
{{k0 → 0.47662, k1 → 0.423432}}
```

Finally, we define a new function as a good approximation of the displacement signal error

```
d2a[m_, SNR_] = Sqrt[(k0 + k1 m^2)/SNR] /. sol1
{Sqrt[(0.47662 + 0.423432 m^2)/SNR]}
```

The approximation of the displacement signal error is

$$\Delta\varepsilon = \sqrt{\frac{0.4766196 + 0.42343225\,\bar{\varepsilon}^2}{SNR}} \tag{5}$$

Figure 6 shows the displacement signal error for *SNR*=10 dB using (5). The maximal error of the approximated value with respect to the exact is 0.0000668, which is sufficiently low because the error of the displacement signal is from the range 0.2 – 0.3.

Figure 7 shows the displacement signal error for $\bar{\varepsilon}$ =0.5 using (5). The maximal error of the approximated value with respect to the exact is 0.000599, which is sufficiently low because the error of the displacement signal is in the range 0.04 – 0.3.

The displacement signal error can be computed using function **`Table`**

```
SNR1 = 10;
m1 = 0;
t2a = Table[{m1, d2a[m1, SNR1]}, {m1, 0, 1, 0.1}]
```

Figures 6 and 7 are obtained using functions **`Table`** and **`Plot`** command

```
ListLinePlot[{t2, t2a}, AxesLabel → {"ε̄", "Δϵ"},
  PlotStyle → {{Black}, {Dashed, Black}, {Dotted, Black}
  PlotRange → {0.21, 0.31}]
```

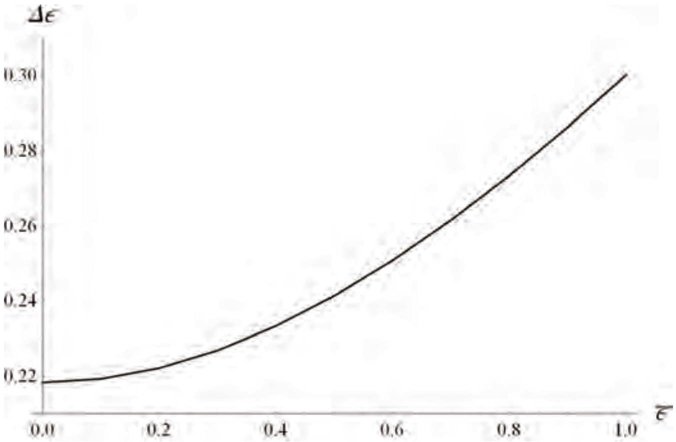

Fig. 6. The error of the displacement signal for *SNR*=10 dB.

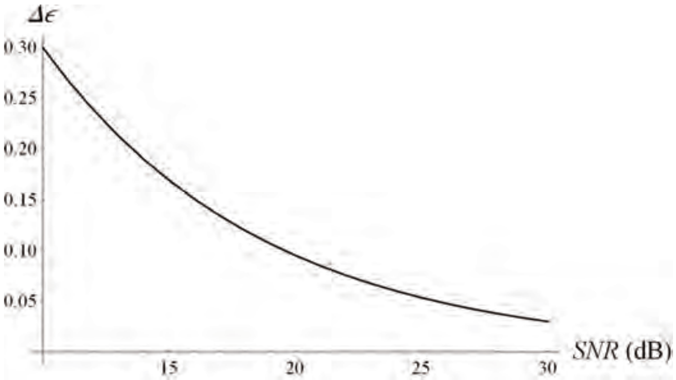

Fig. 7. The error of the displacement signal for $\bar{\varepsilon}$ =0.5.

The differences of numeric calculation using integration and closed form approximation cannot be identified visually in Figures 6 and 7.

V. Conclusion

The displacement signal error of laser illuminated object versus signal-to-noise ratio (SNR) and the mean value of displacement signal ($\bar{\varepsilon}$) is analyzed. The approximation of the displacement signal error is derived in closed form for uncorrelated noise. The displacement signal error is inverse proportional to the root square signal-to-noise ratio, and increases with increase of the mean value of displacement signal. The displacement signal error slightly increases with growth of the mean value of displacement signal.

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