# Design of Novel Two-Level Quantizer with Extended Huffman Coding for Laplacian Source

Zoran Peric[1], Jelena Nikolic[2] and Lazar Velimirovic[3]

***Abstract* – This paper proposes the novel model of two-level scalar quantizer with extended Huffman coding that is designed such that to achieve as close as possible approaching of the bit rate to the source entropy under the given constrain that the SQNR value does not deviate more than 1 dB from the optimal SQNR Lloyd-Max's quantizer value. Unlike to the Lloyd-Max's quantizer, for the proposed quantizer, the asymmetry of representation levels is assumed to provide an unequal probability of representation levels for the symmetric Laplacian probability density function, that in turn provides the proper basis for the further implementation of a lossless compression techniques. The convergence of the proposed quantizer's bit rate to the source entropy is examined in the case of two and three symbol blocks.**

***Keywords* – Scalar quantizer, Extended Huffman coding.**

## I. Introduction

One of the most important steps in the process of converting analog to digital signal is quantization. Discretization of the signal amplitude is done by a quantizer. When developing new models of quantizers it is usually of great importance to find the manner to increase the quality of the quantized signal for a given bit rate or to provide minimization of the bit rate for the desired level of the quantized signal's quality [1] - [3]. The design of a specific quantizer model is simpler and its realization less complex when the required signal quality measured by the Signal to Quantization Noise Ratio (SQNR), is achieved by quantizing signal samples with fewer bits [1] - [3]. In many modern applications, combination of quantizer and lossless coder is used. Most often, quantizer and lossless coder are designed separately, due to simplicity, but obtained performances are not optimal. However, desired performances can be obtained only with joined design of quantizer and lossless coder, which is done in this paper.

In this paper we propose the novel model of scalar quantizer that has a goal to achieve as close as possible approaching of the bit rate to the source entropy under the given constrain that the SQNR value does not deviate more than 1dB from the optimal SQNR Lloyd-Max's quantizer value. In fact, the Lloyd-Max's quantizer [2], [3] presents a special case of the novel quantizer that is proposed in this paper. Unlike the two-level Lloyd-Max's quantizer having the decision threshold settled in zero, the novel quantizer with the same number of quantization levels proposes that the determination of the variable decision threshold is performed in a way that it has a non-negative value, which is designed depending on which SQNR has to be achieved. The basic idea described in this paper is that, unlike to the Lloyd-Max's quantizer, the asymmetry of representation levels is assumed such that to provide an unequal probability of representation levels for the symmetric Laplacian probability density function (PDF). This in turn provides the proper basis for the further implementation of a lossless compression techniques. Among many lossless compression techniques the most suitable for utilization is the extended Huffman coding technique that achieves the lowest average length of code words [3] - [5]. The performance of four types of quantizers with Huffman coding for small and moderate bit rate are analyzed in [6], where it is shown that the best performance is achieved by the hybrid quantizer, composed of the uniform quantizer and the Lloyd-Max's quantizer when Huffman coding technique is applied. Due to the efficient initialization problem of the Lloyd-Max's quantizer's algorithm and the high design complexity of the Lloyd-Max's quantizer with a large number of quantization levels [7], as well as due to the lack of an effective implementation of the Huffman coding technique on the quantizers with large number of quantization levels [1], [4] - [6], we propose the quantizer having only two representation levels. As with the Lloyd-Max's quantizer these representation levels are determined from the centroid condition. The design procedure of the scalar quantizer having the representation levels also determined in accordance with the centroid condition for the Laplacian and Gaussian source is given in [8] along with the analysis of the absolute and the mean-square error distortion for the low bit rate.

This paper is organized as follows. After brief introduction, the novel quantizer with variable decision threshold is described in Section 2. In Sections 3 and 4 a detailed description of the proposed quantizer's code book determination along with the formulation of the extended Huffman coding is provided. The achieved numerical results for the proposed quantizer with extended Huffman coding and the Laplacian source are the topics addressed in Section 5. Additionally, Section 5 is devoted to the conclusions which summarize the contribution achieved in the paper.

## II. Design of Two-Level Quantizer with Variable Decision Threshold

This section contains a detailed description of the novel quantizer model having variable decision threshold. Quantization is the process of replacing analog samples with

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the nearest allowed value from a discrete set of $N$ amplitude values [2], [3]. An $N$ - level scalar quantizer $Q$ is defined by mapping $Q$: $R \to Y$, where $R$ is the set of real numbers, and

$$Y \equiv (y_1, y_2, y_3, \ldots, y_N) \subset R \tag{1}$$

is a set of representation levels that makes the code book of size $|Y| = N$ [2], [3]. Associated with every $N$ - level scalar quantizer is partition of the set of real numbers into $N$ cells $R_i = (t_{i-1}, t_i]$, $i = 1, \ldots, N$, where $t_i$, $i = 0, 1, \ldots, N$ are decision thresholds and it holds $Q(x) = y_i$, $x \in R_i$. The quantizer designed iteratively in accordance with the centroid condition and the condition of the nearest neighbor is the optimal Lloyd-Max's quantizer [2], [3]. In other words, for a given PDF $p(x)$ of the input signal of variance $\sigma^2$ and for the considered number of quantization levels $N$, the minimum value of distortion, i.e. the maximum value of SQNR is achieved by using the Lloyd-Max's quantizer. The quantizer we propose in this paper is defined by the variable decision threshold along with the two representation levels (see Fig. 1) that are, as for the Lloyd-Max's quantizer, determined from the centroid condition. Particularly, this variable decision threshold we determine depending on the quality, measured by SQNR that has to be achieved. Note that in the special case, when the mentioned variable decision threshold has zero value, the proposed quantizer becomes optimal. For the assumed Laplacian PDF of the unit variance [2], [3]

$$p(x) = \frac{1}{\sqrt{2}} \exp\left(-\sqrt{2}|x|\right), \tag{2}$$

the representation levels of the proposed quantizer are determined as follows

$$y_1 = \frac{\int_{-\infty}^{t_1} x p(x) dx}{\int_{-\infty}^{t_1} p(x) dx} = \frac{\sqrt{2} + 2t_1}{2 - 4\exp\left(\sqrt{2}t_1\right)}, \tag{3}$$

$$y_2 = \frac{\int_{t_1}^{\infty} x p(x) dx}{\int_{t_1}^{\infty} p(x) dx} = t_1 + \frac{1}{\sqrt{2}}, \tag{4}$$

where the variable decision threshold is denoted by $t_1$. From the last two equations it is obvious that the representation levels of the proposed quantizer are not symmetrical.

The performances of the quantizer are often determined by SQNR which is defined as follows [2], [3]

$$\text{SQNR} = 10 \log\left(\frac{\sigma^2}{D}\right) \tag{5}$$

and expressed in dB where $\sigma^2$ is the variance of the input signal $x$, while $D$ is the distortion added with quatization. Assuming the unit variance for the given range of SQNR values one can firstly determine the appropriate $D$ values

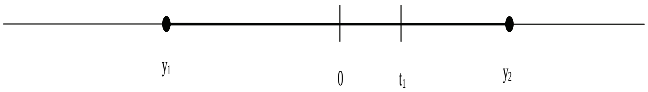


Fig. 1. Model of a two-level quantizer with asymmetric representation levels

$$D = \frac{\sigma^2}{10^{\frac{\text{SQNR}}{10}}} = \frac{1}{10^{\frac{\text{SQNR}}{10}}}. \tag{6}$$

Further defining the distortion of the proposed quantizer

$$D = \int_{-\infty}^{t_1} (x - y_1)^2 p(x) dx + \int_{t_1}^{\infty} (x - y_2)^2 p(x) dx \tag{7}$$

and by combining with Eqs. 3 and 4, a closed form expression for the distortion of the proposed quantizer is derived as a function of the variable decision threshold $t_1$

$$D = \frac{3 - 4\exp\left(\sqrt{2}t_1\right) + 2\sqrt{2}t_1 + 2t_1^2}{2 - 4\exp\left(\sqrt{2}t_1\right)}. \tag{8}$$

Using the last expression and the distortion values obtained from the Eq. 6 for the considered range of SQNR values, the appropriate value of the threshold $t_1$ can be determined, and hence, the design of the proposed quantizer is enabled.

## III. Extended Huffman Coding for Two Symbol Sources

In this section a basic concept of the very popular lossless compression technique, called extended Huffman coding is presented. The procedure of Huffman coding includes determining the optimal length of code words for a given probability of symbols [1], [3], [5], [9]. Note that it is sometimes beneficial to additionally reduce the bit rate by blocking more than one symbol together. In the mentioned cases, the extended Huffman coding technique is used. Particularly, the extended Huffman coding is the procedure of determining the optimal length of code words for blocks of two or more symbols [3], [5], [9].

Let us denote by $p_1$ the probability that sample of the input signal has a lower value than the value of decision threshold $t_1$

$$p_1 = \int_{-\infty}^{t_1} p(x) dx = 1 - \frac{1}{2}\exp(-\sqrt{2}t_1), \tag{9}$$

and by $p_2$, the probability that sample of the iput signal has a greater value than the value of decision threshold $t_1$

$$p_2 = \int_{t_1}^{\infty} p(x) dx = \frac{1}{2}\exp(-\sqrt{2}t_1). \tag{10}$$

These probabilities actually refer to the symbol probabilities, i.e. to the probabilities of occurence of representation levels $y_1$ and $y_2$. Since we consider two-level quantizer we in fact observe two symbol source. As the extended Huffman coding procedure blocks more than one symbol together, we can now define probabilities of two and three symbol blocks as

$$P_{i,j} = p_i p_j, \quad i = 1,2, \ j = 1,2, \tag{11}$$

$$P_{i,j,k} = p_i p_j p_k, \quad i = 1,2, \ j = 1,2, k = 1,2. \tag{12}$$

Note that blocking two symbols together means the symbol alphabet size goes from $m$ to $m^2$, where $m$ is the size of the initial symbol alphabet. In this paper, we consider two cases, of two and three symbol blocks, such that the size of the extended alphabet is 4 and 8, respectively. For the proposed quantizer with extended Huffman coding we examine the convergence of the bit rate to the source entropy. Source entropy for two and three symbol blocks are given by the following expressions, respectively [9]

$$H = \sum_{i=1}^{2}\sum_{j=1}^{2} P_{i,j} ld \frac{1}{P_{i,j}}, \tag{13}$$

$$H = \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} P_{i,j,k} ld \frac{1}{P_{i,j,k}}. \tag{14}$$

The average bit rate of the observed quantizer in the case of two symbol blocks can be determined as

$$\overline{R} = \sum_{i=1}^{2}\sum_{j=1}^{2} P_{i,j} l_{i,j}, \tag{15}$$

where $l_{i,j}$, $i = 1, 2$, $j = 1, 2$ stands for the length of the code words. Similarly, the average bit rate in the case of three symbol blocks is determined by

$$\overline{R} = \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} P_{i,j,k} l_{i,j,k}. \tag{16}$$

The procedure of determining the length of the code words using the extended Huffman coding and the code book construction is consisted of the following steps:

**Step 1.** Determining the symbol block probabilities, further sorting in descending order and finally assigning appropriate probabilities to the initial nodes of the graph.

**Step 2.** Application of an iterative process, where in each iteration the connection of the two nodes with the smallest probabilities is done and the sum of their probabilities is assigned to a new node. Processing further until the nodes' sum of the probabilities joining in the last step becomes equal to one (see Fig. 2).

**Step 3.** The construction of code words. Code word for each symbol is determined by beginning from the tree root (node with probability 1) and branches, to which the allocation of zero value is acquired (upper branch) and 1 (lower branch). Assignment process continues to the left until all possible branches are covered. Code word is formed from zeros and ones that are on the path from the root to the node that corresponds to that symbol (see Fig. 2)

An example of extended Huffman coding is illustrated in Fig. 2 for the values of symbol probabilities $p_1 = 0.8958$ and $p_2 = 0.1041$, and for the size of the symbol blocks of $M = 3$. In this case, there are 8 probability symbol blocks $P_{i,j,k}$, $i = 1, 2$, $j = 1, 2$, $k = 1, 2$, having the values given in Table I. The result of the extended Huffman coding procedure is an extended Huffman codebook which for the illustrated example is also given in Table I. From the Table I one can notice that the extended Huffman coding procedure creates variable length codes, where higher probability symbol blocks are coded by shorter codes.

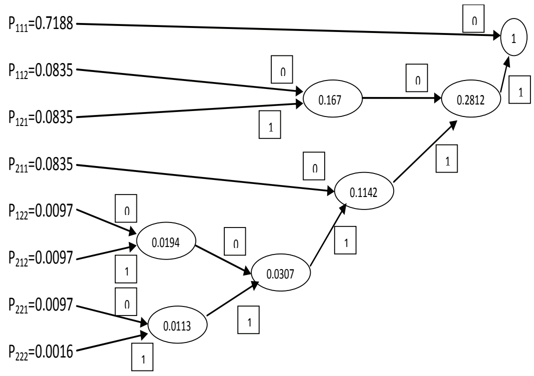


Fig. 2. Example of extended Huffman code construction: forming the tree and assigning the code words

TABLE I
EXTENDED HUFFMAN CODEBOOK EXAMPLE

| $i,j,k$ | $P_{i,j,k}$ | Extended Huffman code | $l_{i,j,k}$ |
|---|---|---|---|
| 1,1,1 | 0.7188 | 0 | 1 |
| 1,1,2 | 0.0835 | 100 | 3 |
| 1,2,1 | 0.0835 | 101 | 3 |
| 1,2,2 | 0.0097 | 11100 | 5 |
| 2,1,1 | 0.0835 | 110 | 3 |
| 2,1,2 | 0.0097 | 11101 | 5 |
| 2,2,1 | 0.0097 | 11110 | 5 |
| 2,2,2 | 0.0016 | 11111 | 5 |

## IV. ALGORITHM OF DESIGNING NOVEL QUANTIZER WITH VARIABLE DECISION THRESHOLD AND EXTENDED HUFFMAN CODING

The determination of the proposed quantizer performance, for a given signal quality, i.e. the desired level of SQNR, consists of the following steps:

**Step 1.** The determination of the distortion $D$ by using Eq. 6 for the desired quality of the quantized signal, i.e. for an assumed value of SQNR.

**Step 2.** The design of the decision threshold $t_1$ according to Eq. 8 for $D$ calculated in step 1.

**Step 3.** Determining the representation levels $y_1$ and $y_2$ (Eqs. 3 and 4), the probabilities $p_1$ and $p_2$ (Eqs. 9 and 10), for the value of the decision threshold $t_1$ calculated in the previous step.

**Step 4.** Determining the symbol block probabilities $P_{i,j}$, $i = 1, 2$, $j = 1, 2$ (Eq. 11) and $P_{i,j,k}$, $i = 1, 2$, $j = 1, 2$, $k = 1, 2$ (Eq. 12) using the probabilities $p_1$ and $p_2$.

**Step 5.** The determination of the code word lengths by using the extended Huffman coding, $l_{i,j}$, $i = 1, 2$, $j = 1, 2$ (in the case of two symbol blocks), $l_{i,j,k}$, $i = 1, 2$, $j = 1, 2$, $k = 1, 2$ (in the case of three symbol blocks).

**Step 6.** Calculating the source entropy $H$ (Eqs. 13 and 14) and the average bit rate $\overline{R}$ (Eqs. 15 and 16).

## V. Numerical Results and Conclusion

Numerical results presented in this section for the proposed two-level quantizer with extended Huffman coding are obtained for the cases where the SQNR value does not deviate more than 1 dB from the optimal quantizer SQNR value with the same number of quantization levels. The optimal SQNR value of the Lloyd-Max's quantizer having two quantization levels is 3 dB, which means that the SQNR range in which we consider the performance of the proposed quantizer is [2 dB , 3 dB]. The calculated performance of the proposed quantizer in the case of two and three symbol blocks are shown in Fig. 3. Particularly, Fig. 3 shows the dependence of the average bit rate $\overline{R}$ and the source entropy $H$ on the distortion for the proposed quantizer in the case of the extended Huffman coding applied on two and three symbol blocks. One can notice that the $\overline{R}$ of the proposed quantizer approaches the source entropy $H$ where this convergence is greater in the case of three symbol blocks than in the case of two symbol blocks. By blocking more and more symbols together, the size of the alphabet exponentialy grows, and the extended Huffman coding technique becomes impractical [9]. Accordingly, in this paper our analysis is constrained to the case of three symbol blocks. Table II contains the values of $\overline{R}$ for two and three symbol blocks, distortion, decision thresholds, as well as the values of probabilities $p_1$ and $p_2$ that for the considered range of SQNR are achieved by the proposed quantizer. From the results given in Table II and Fig. 3 one can observe that when the SQNR value deviats up to 0.5 dB from the optimal SQNR value, there is a little deviation of $\overline{R}$ from $H$ in the case of three symbol blocks. However, when the mentioned deviation of SQNR is in the range of 0.5 dB to 1 dB, a slightly larger deviation of $\overline{R}$ from $H$ can be perceived. Observe that in both ranges the increasing convergence of $\overline{R}$ to $H$ is achieved in the case of three symbol blocks. It is important to notice that for the proposed quantizer in the case of three symbol blocks with an average bit rate reduction of 0.35 bits, the reduction in SQNR of 0.5 dB is achieved. This is about 0.9 dB smaller SQNR reduction for the same amount of the compression than the one ascertained in the considered range of $\overline{R}$ [9]. Hence, it is obvious that the proposed quantizer represents a very efficient coding solution. Finally, from the last row in Table II one can notice that the optimal Lloyd-Max's quantizer is actually the special case of the proposed quantizer. Particularly, when the decision threshold $t_1$ of the proposed quantizers is settled to zero, the proposed quantizer is Lloyd-Max's quantizer that has the symmetrical representation levels, i.e. equal probabilities $p_1$ and $p_2$. For such values of probabilities, the values of the $H$ and $\overline{R}$ of the proposed quantizer are equal and amount to one.

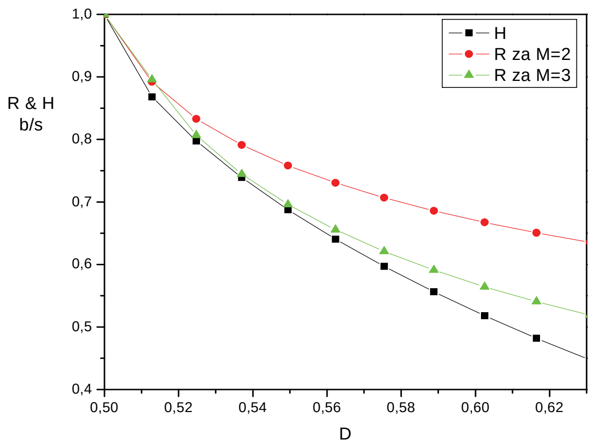


Fig. 3. The dependency of the bit rate and the entropy on the distortion for the proposed quantizer

TABLE II
PERFORMANCE OF THE PROPOSED QUANTIZER IN THE CASE OF TWO AND THREE SYMBOL BLOCKS

| *SQNR* | *D* | *H* | $t_1$ | $p_1$ | $p_2$ | $\overline{R}$ $M = 2$ | $\overline{R}$ $M = 3$ |
|---|---|---|---|---|---|---|---|
| 2 | 0.631 | 0.447 | 1.188 | 0.906 | 0.093 | 0.635 | 0.519 |
| 2.1 | 0.617 | 0.482 | 1.109 | 0.895 | 0.104 | 0.650 | 0.540 |
| 2.2 | 0.603 | 0.518 | 1.032 | 0.883 | 0.116 | 0.667 | 0.564 |
| 2.3 | 0.589 | 0.556 | 0.955 | 0.870 | 0.129 | 0.685 | 0.590 |
| 2.4 | 0.575 | 0.597 | 0.876 | 0.855 | 0.144 | 0.706 | 0.620 |
| 2.5 | 0.562 | 0.641 | 0.794 | 0.837 | 0.162 | 0.730 | 0.655 |
| 2.6 | 0.549 | 0.688 | 0.709 | 0.816 | 0.183 | 0.758 | 0.695 |
| 2.7 | 0.537 | 0.739 | 0.618 | 0.791 | 0.208 | 0.791 | 0.744 |
| 2.8 | 0.525 | 0.797 | 0.515 | 0.758 | 0.241 | 0.832 | 0.806 |
| 2.9 | 0.513 | 0.868 | 0.387 | 0.710 | 0.289 | 0.892 | 0.895 |
| 3 | 0.5 | 1 | 0 | 0.5 | 0.5 | 1 | 1 |

In this paper a novel class of quantizers having variable decision thresholds with extended Huffman coding is presented. Based on the proposed quantizer analysis, it is shown that by using the extended Huffman coding technique and the set of quantizers with variable decision thresholds, approaching of the average bit rate to the source entropy can be achieved.

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