Spectrum Optimization of Truncated Complex Hadamard Transform

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Abstract – **An algorithm for spectrum optimization of truncated Complex Hadamard Transform on the base of minimization of mean-squared error of reconstructed transform coefficients is presented. The developed algorithm is simulated on Matlab 6.5 environment and the obtained results showed increasing of signal to noise ratio with about 0.5 dB.**

Keywords – **Digital Signal Processing, Complex Hadamard Transform, Data Compression, Orthogonal Transforms.**

I. Introduction

Data compression, the art and science of reducing the amount of data required to represents 1D and 2D information, is one of the most useful and commercially successful technologies in the field of digital signal processing [1]. Dimensionality reduction in computation is a major signal processing application. One of the common compression approaches for reducing of spatial and temporal redundancy of information is based on block transform coding (BTC) [1]-[4], in which, a reversible, linear transform is used to map each block into a set of transform coefficients. Discrete orthogonal (unitary) transforms [3], [4], used in BTC, have found applications in many areas of N-dimensional signal processing, spectral analysis, pattern recognition, digital coding, computational mathematic and etc. Stated simply, these transform coefficients that are small may be excluded from processing operations, such as filtering, without much loss in processing accuracy.

The discrete integer Walsh-Hadamard Transform (WHT) is a fairly simple orthogonal transform and is an example of a generalized class of Fourier transforms [3]. The idea of using complex, rather than integer transforms matrices for spectral processing, analysis and watermarking has been shown in [5], [6], [7] and [8]. From the Complex Hadamard Transform (CHT), several complex decisions diagrams are derived and analysis of more general CHT properties for 1D and 2D signals are investigated [9],[10].

In this paper an algorithm for optimization of reduced spectrum of Complex Hadamard Transform is developed, using minimization of mean-squared error of reconstructed one- and two-dimensional signals. The obtained results showed increasing of signal to noise ratio with about 0.3 – 0.7 dB for any unitary transform.

The developed optimization algorithm is simulated on Matlab 6.5 environment for one test image "Lena" and the results of four unitary transforms – FFT, DCT, WHT and CHT are given.

II. Mathematical Description

Using the basic forward and inverse one-dimensional complex Hadamard transform for N coefficients [9],[11], the input signal vector $\vec{A}=[a_0,a_1,a_2,......,a_{N-1}]$ and the output spectral vector - $\vec{B}=[b_0,b_1,b_2,......,b_{N-1}]$ are joined by the equations:

$$\vec{B}=\frac{1}{N}[CH]\vec{A} \ , \ \vec{A}=\frac{1}{N}[CH]^t\vec{B} . \qquad (1)$$

The discrete forward and the inverse orthogonal transform can be written by the following:

$$\left| \begin{array}{ll} b_j=\frac{1}{N}\sum_{i=0}^{N-1}a_i t_{ij} , & \text{for}: j=\overline{0,N-1} \\ a_i=\sum_{j=0}^{N-1}b_j t_{ij} & , \text{for}: \ i=\overline{0,N-1} \end{array} \right. , \qquad (2)$$

where: t_{ij} are transform coefficients.

The goal of the transformation process is to decorrelate the values of each sub-block, or to pack as much information as possible into the smallest number of spectral coefficients. The quantization stage than selectively eliminates or more coarsely quantizes the coefficients that carry the least amount of information in a predefined sense. These coefficients have the smallest impact on reconstructed sub-block quality. In more cases the first k important coefficients are saved and the next N-k-1 coefficients are truncated.

The inverse decomposition from equation (2) can be written by the following:

$$a_i=\sum_{j=0}^{N-1}b_j t_{ij}=\sum_{j=0}^{k}b_j t_{ij}+\sum_{j=k+1}^{N-1}b_j t_{ij} \quad , \text{for}: \ i=\overline{0,N-1} . \qquad (3)$$

The last N-k-1 coefficients can be substituted by the value A and the approximated signal can be obtained from the truncated expansion:

$$\hat{a}_i=\sum_{j=0}^{N-1}b_j t_{ij}=\sum_{j=0}^{k}b_j t_{ij}+A\sum_{j=k+1}^{N-1}t_{ij} \quad . \qquad (4)$$

The difference between the input signal and its approximation can be given with the equation (5):

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$$\Delta a_i = a_i - \hat{a}_i = \sum_{j=0}^{N-1} b_j t_{ij} - \sum_{j=0}^{k} b_j t_{ij} - A \sum_{j=k+1}^{N-1} t_{ij} = \sum_{j=k+1}^{N-1} (b_j - A) t_{ij} \quad (5)$$

The mean-square error then is:

$$\begin{aligned} \overline{\varepsilon^2} &= \frac{1}{N} \sum_{i=0}^{N-1} (a_i - \hat{a}_i)^2 = \frac{1}{N} \sum_{i=0}^{N-1} \left[\sum_{j=k+1}^{N-1} (b_j - A) t_{ij} \right]^2 = \\ &= \frac{1}{N} \sum_{i=0}^{N-1} \left[\sum_{s=k+1}^{N-1} (b_s - A) t_{is} \right] \left[\sum_{l=k+1}^{N-1} (b_l - A) t_{il} \right] = \\ &= \frac{1}{N} \sum_{i=0}^{N-1} \left[\sum_{s=k+1}^{N-1} \sum_{l=k+1}^{N-1} (b_s - A)(b_l - A) t_{is} t_{il} \right] = \\ &= \sum_{j=k+1}^{N-1} (b_j - A)^2 \end{aligned} \quad (6)$$

where: $t_{is} t_{il} = \begin{cases} 1 \text{ ,for } s = l \\ 0 \text{ ,for } s \neq l \end{cases}$ is true for each orthogonal transform.

The minimum of mean-square error is obtained by differentiation:

$$\frac{\partial \overline{\varepsilon^2}}{\partial A} = -2 \sum_{j=k+1}^{N-1} (b_j - A) = 0 \ , \quad (7)$$

and the optimum value for truncated coefficients is:

$$A_{opt} = \frac{1}{N - k - 1} \sum_{j=k+1}^{N-1} b_j = m_b \ . \quad (8)$$

The equation (8) show that the mean-square error (MSE) decreases for any orthogonal transform via approximation of reduced spectral coefficients by average of their values.

The improvement of MSE can be calculated from equations (6) and (8):

$$\begin{aligned} \Delta \overline{\varepsilon^2} &= \overline{\varepsilon^2} - \overline{\varepsilon^2_{min}} = \sum_{j=k+1}^{N-1} b_j^2 - \sum_{j=k+1}^{N-1} (b_j - m_b)^2 = \\ &= \sum_{j=k+1}^{N-1} b_j^2 - \sum_{j=k+1}^{N-1} \left[b_j^2 - 2 b_j m_b + m_b^2 \right] = \\ &= \sum_{j=k+1}^{N-1} \left[2 b_j m_b - m_b^2 \right] \end{aligned} \quad , \quad (9)$$

where: $\overline{\varepsilon^2} = \sum_{j=k+1}^{N-1} b_j^2$ is MSE for zero approximation of truncated coefficients and $\overline{\varepsilon^2_{min}} = \sum_{j=k+1}^{N-1} (b_j - m_b)^2$ is MSE for mean approximation of truncated coefficients.

This algorithm can be summarized for 2D signals (images). The forward and the inverse discrete transform of sub-image *g(x,y)* of size *N*x*N* can be expressed as the following equations:

$$\begin{aligned} S(u,v) &= \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} g(x,y).r(x,y,u,v) \\ g(x,y) &= \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} S(u,v).t(x,y,u,v) \end{aligned} \quad , \quad (10)$$

In this equations *r(x,y,u,v)* and *t(x,y,u,v)* are called the forward and inverse transformation kernels, respectively. Because the inverse kernel *t(x,y,u,v)* in (10) depends only on the indices *(x,y,u,v)* and not on the values of *g(x,y)* and *S(u,v)*, it can be viewed as defining a set of basis functions or basis images.

This interpretation becomes clearer if the equation is modified in matrix form:

$$\mathbf{G} = \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} S(u,v) \mathbf{T}_{uv} \ , \quad (11)$$

where: **G** is *N*x*N* matrix containing the pixels of *g(x,y)*, the matrices $\mathbf{T}_{uv}$ are the basis images and S(u,v) are the spectral coefficients.

We can define a transform coefficients masking function:

$$\chi(u,v) = \begin{cases} 0, \text{if } S(u,v) \text{ satisfies a specified truncation criterion} \\ 1, \text{otherwise.} \end{cases}$$

An approximation of **G** can be obtained from the truncated expansion:

$$\begin{aligned} \hat{\mathbf{G}} = \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \chi(u,v) S(u,v) \mathbf{T}_{uv} &= \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \chi(u,v) S(u,v) \mathbf{T}_{uv} - \\ &- \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \left[1 - \chi(u,v) \right] A \mathbf{T}_{uv} \end{aligned} \quad ,(12)$$

The mean-square error between sub-image **G** and its approximation $\hat{\mathbf{G}}$ then is:

$$\overline{\varepsilon^2} = E\left\{ \left\| \mathrm{G} - \hat{\mathrm{G}} \right\|^2 \right\} , \quad (13)$$

$$\begin{aligned} \overline{\varepsilon^2} &= E\left\{ \left\| \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} S(u,v) \mathbf{T}_{uv} - \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \chi(u,v) S(u,v) \mathbf{T}_{uv} \right\|^2 \right\} = \\ &= E\left\{ \left\| \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \left[1 - \chi(u,v) \right] S(u,v) \mathbf{T}_{uv} - \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \left[1 - \chi(u,v) \right] A \mathbf{T}_{uv} \right\|^2 \right\} \\ &= E\left\{ \left\| \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \left[1 - \chi(u,v) \right] \left[S(u,v) - A \right] \mathbf{T}_{uv} \right\|^2 \right\} \\ &= \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \left[1 - \chi(u,v) \right] \left[S(u,v) - A \right]^2 \end{aligned}$$

The minimum of mean-square error is obtained by the following:

$$\frac{\partial \overline{\varepsilon^2}}{\partial A} = -2 \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \left[1 - \chi(u,v) \right] \left[S(u,v) - A \right] = 0 \, , \quad (14)$$

and the optimum value for the truncated coefficients is:

$$A_{opt} = \frac{\sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \left[1 - \chi(u,v) \right] S(u,v)}{\sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \left[1 - \chi(u,v) \right]} \ . \quad (15)$$

III. Experimental Results

The developed optimization algorithm is simulated on Matlab 6.5 environment for four 2D unitary transforms – Discrete Fourier Transform, Discrete Cosine Transform, Discrete Walsh-Hadamard Transform and Complex Hadamard Transform. The obtained experimental results for the test image “Lenna” (512x512, 8 bits) with sub-image kernel 8x8 are given on Table 1.

TABLE 1.

Reduction type	MSE	NMSE	SNR, dB	PSNR, dB
48 Reduced Coefficients, 16 saved				
Zero FFT	107.383	$8.65x10^{-6}$	50.6263	27.8554
Mean FFT	97.1544	$7.83x10^{-6}$	51.0610	28.2902
Zero DCT	0.5078	$4.09x10^{-8}$	73.8785	51.1076
Mean DCT	0.6590	$5.31x10^{-8}$	72.7471	49.9762
Zero WHT	19.6889	$1.58x10^{-6}$	57.9934	35.2226
Mean WHT	16.7618	$1.35x10^{-6}$	58.6924	35.9216
Zero CHT	16.0925	$1.30x10^{-6}$	58.8694	36.0985
Mean CHT	15.2343	$1.23x10^{-6}$	59.1074	36.3365
55 Reduced Coefficients, 9 saved				
Zero FFT	116.777	$9.41\ x10^{-6}$	50.2620	27.4912
Mean FFT	106.358	$8.57\ x10^{-6}$	50.6679	27.8971
Zero DCT	0.6828	$5.50\ x10^{-8}$	72.5924	49.8216
Mean DCT	0.5800	$4.67\ x10^{-8}$	73.3013	50.5304
Zero WHT	43.6631	$3.52\ x10^{-6}$	54.5345	31.7636
Mean WHT	42.5557	$3.43\ x10^{-6}$	54.6461	31.8752
Zero CHT	46.4509	$3.74\ x10^{-6}$	54.2657	31.4948
Mean CHT	42.2766	$3.40\ x10^{-6}$	54.6747	31.9038

In the first part of the table the results for 48 reduced coefficients (4x4 are saved) approximated with zero and mean values are shown, and in the second part the same experiments for 55 reduced coefficients (3x3 - saved) are shown. The calculated values for the mean-square error (MSE), normalized mean-square error (NMSE), signal to noise ratio (SNR) and peak signal to noise ratio (PSNR) showed increasing of each parameter with about 0.5 %. The input image is shown on Fig.1, and the output images for 48 reduced coefficients are showed on Fig.2.

Fig.1. Input test image “Lena” (512x512, 8 bits)

IV. CONCLUSION

Method for spectrum optimization of truncated orthogonal transforms is presented. The improvement of quality by the compression of one-dimensional and two-dimensional signals is theoretically proved. An algorithm for block truncation coding is developed on the base of minimization of mean-squared error of reduced spectral coefficients reconstruction. The experimental results with four transformations are given.

The main advantages of the developed algorithm are:

- increasing the signal to noise ratio by the compression with truncated discrete orthogonal transforms of one- and two-dimensional signals;
- decreasing the preserved coefficients and increasing compression ratio by the using any discrete orthogonal transforms;
- using the CHT instead most complicated Fourier transform and keep the possibilities for working with complex spectrum.

The presented spectrum optimization for discrete orthogonal transforms can be used in digital signal processing for spectral analysis, pattern recognition, digital watermarking, transformation, coding and transmission of one-dimensional and two-dimensional signals.

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Fig.2. Output images after reduction with 48 coefficients for FFT, DCT, WHT and CHT