The Concept of Quasi Orthogonality Applied in Technical Systems

Dragan Antić[1], Marko Milojković[2], Saša Nikolić[3], Darko Mitić[4] and Staniša Perić[5]

Abstract – **In this paper we introduce quasi-almost orthogonal filters, as a new class of filters, based on Legendre type quasi-almost orthogonal polynomials. These filters are generators of quasi-almost orthogonal signals for which we have derived and presented important properties and relations. Our work is based on classical theory of orthogonality, and orthogonal rational functions, and also on new results in this field of research. Quasi-almost orthogonal filters can be successfully used for signal approximation as well as for modeling, identification, and analysis of dynamical systems. Based on mathematical results, we have designed schemes for practical realization of these filters.**

Keywords – **Quasi orthogonality, Almost orthogonality, Quasi-almost orthogonal filters.**

I. Introduction

Orthogonal polynomials have been in the focus of the research for the last two centuries. One of the most important applications of orthogonal polynomials is designing orthogonal filters. Today these filters provide an efficient tool for identification, modeling and control of dynamical systems.

Concept of quasi orthogonality is introduced for the first time in 1923 [1] as a tool for solving the problem of moments in mechanics. Quasi orthogonal functions and especially quasi orthogonal polynomials as well as numerous applications are discussed in many papers [2-5]. It is important to notice that classical orthogonal filters and orthogonal signal generators have transfer functions with the order of numerator polynomial for one less then denominator. In practice there is often need for filters of more general type i.e., filters with difference in orders of transfer functions polynomials higher then one. This can be accomplished by using quasi orthogonal filters.

The components that are used for designing any real (practical) system are not perfect and their parameters values vary in the range of allowed tolerance. The reasons can be various: imperfect manufacturing, systems exploitation conditions (environment temperature, pressure, moisture, electromagnetic fields, variations in voltage)… With respect to that fact, every real system is in some way imperfect, so the models of these systems should reflect this fact. Almost orthogonal filters designed in earlier papers [6-8] can be successfully used for modeling and analysis of the systems with imperfections.

In this paper we consider quasi-almost Legendre type orthogonal polynomials. Based on these polynomials, we have designed first order quasi-almost Legendre type orthogonal filter. For the first time, here, we combine two theories into new class of filters. These filters can be used as generators of functions sequence, suitable for modeling and analysis of appropriate imperfect systems that can be described with transfer function with difference in polynomials orders higher that one.

II. Almost Orthogonal Polynomials – A New Approach

Our design of orthogonal filters, is based on shifted Legendre polynomials orthogonal over interval (0, 1). On the other side, technical systems operate in the real time, so we need the corresponding approximation over interval (0, ∞). The solution is to use the substitution $x = e^{-t}$. In this manner polynomial sequence orthogonal over (0, 1), become exponential polynomial sequence orthogonal over interval (0, ∞).

Consider the orthogonal Legendre polynomials in their explicit form:

$$P_n(x) = \sum_{i=0}^{n} A_{n.i} x^i \tag{1}$$

where:

$$A_{n.i} = \frac{1}{n!}(-1)^{n-i}\binom{n}{i}\frac{(n+i)!}{i!} \tag{2}$$

The first few members of the sequence are:

$$\begin{aligned} &P_0(x) = 1, \\ &P_1(x) = 2x - 1, \\ &P_2(x) = 6x^2 - 6x + 1, \ldots \end{aligned} \tag{3}$$

These polynomials are orthogonal over interval (0, 1), with weight function $w(x)$=1, i.e.:

[1]Dragan Antić is with the University of Niš, Faculty of Electronic Engineering, Aleksandra Medvedeva 14, 18000 Niš, Serbia, E-mail: dragan.antic@elfak.ni.ac.rs

[2]Marko Milojković is with the University of Niš, Faculty of Electronic Engineering, Aleksandra Medvedeva 14, 18000 Niš, Serbia, E-mail: marko.milojkovic@elfak.ni.ac.rs

[3]Saša Nikolić is with the University of Niš, Faculty of Electronic Engineering, Aleksandra Medvedeva 14, 18000 Niš, Serbia, E-mail: sasa.s.nikolic@elfak.ni.ac.rs

[4]Darko Mitić is with the University of Niš, Faculty of Electronic Engineering, Aleksandra Medvedeva 14, 18000 Niš, Serbia, E-mail: darko.mitic@elfak.ni.ac.rs

[5]Staniša Perić is with the University of Niš, Faculty of Electronic Engineering, Aleksandra Medvedeva 14, 18000 Niš, Serbia, E-mail: stanisa.peric@elfak.ni.ac.rs

$$\int_0^1 P_m(x)P_n(x)dx = \begin{cases} 0, & m \neq n \\ N_i, & m = n \end{cases} \tag{4}$$

and they can be successively used for modeling [9], and control [10] of dynamical systems as well as for identification of specific systems parameters [11].

Corresponding almost orthogonal polynomials $P_n^{(\varepsilon)}(x)$ can be defined as [6]:

$$\int_0^1 P_m^{(\varepsilon)}(x)P_n^{(\varepsilon)}(x)dx = \begin{cases} \varepsilon, & m \neq n \\ N_n^{(\varepsilon)}, & m = n \end{cases} \tag{5}$$

where ε represent a very small positive constant $(0 < \varepsilon << 1)$.

The connection between classical orthogonal and almost orthogonal polynomials is proved in [7] with following relation:

$$P_n^{(\varepsilon)}(x) = P_n(x) + \sum_{k=1}^{n-1} \frac{b_k}{\|P_k\|^2} P_k(x) \tag{6}$$

where $\|P_k\|^2$ represents the square of the norm and b_k are polynomials dependent on ε.

The first few members of the Legendre almost orthogonal polynomial sequence are:

$$\begin{aligned} &P_0^{(\varepsilon)}(x) = 1, \\ &P_1^{(\varepsilon)}(x) = 2x - (1-2\varepsilon), \\ &P_2^{(\varepsilon)}(x) = 6x^2 - 6\left(1-12\varepsilon+12\varepsilon^2\right)x + \left(1-30\varepsilon+36\varepsilon^2\right), \ldots \end{aligned} \tag{7}$$

Complete mathematical background for designing almost orthogonal filters based on polynomials given by Eq. (6) can be found in [6, 7]. These filters represent generalization of the Legendre type orthogonal filters [7, 10]. Obviously, for ε=0 almost orthogonal filter becomes strictly orthogonal filter. Practical realization of almost orthogonal filters defined in such way is elaborated in [8].

In this paper, we will define almost orthogonality in different manner in order to accomplish further simplifications in filters design. First, we define almost orthogonal Legendre polynomials $P_n^{\delta}(x)$:

$$P_n^{\delta}(x) = \sum_{i=0}^{n} A_{n,i}^{\delta} x^i \tag{8}$$

where:

$$A_{n,i}^{\delta} = (-1)^{n+i} \frac{\Gamma(n\delta + i + 1)}{\Gamma(n\delta + 1)\, i!\,(n-i)!} \tag{9}$$

In Eq. (9), δ represents constant: $\delta=1+\varepsilon\approx1$ and Γ represents gamma function.

Now, we can define almost orthogonality in the other way:

$$\int_0^1 P_m^{\delta}(x)P_n^{\delta}(x)dx = \begin{cases} (1-\delta)\sum_{k=1}^{n} kQ_k, & m \neq n \\ N_n^{\delta}, & m = n \end{cases} \tag{10}$$

where:

$$Q_k = \frac{\prod_{j=1}^{m}(k - j\delta)\prod_{j=1}^{n}(k + j\delta)}{k^2\prod_{j=1}^{m}(k+j)\prod_{j=1}^{n}(k-j)} \tag{11}$$

The first few members of the Legendre almost orthogonal polynomial sequence defined with Eq. (8) are:

$$\begin{aligned} &P_0^{\delta}(x) = 1, \\ &P_1^{\delta}(x) = (\delta+1)x - \delta, \\ &P_2^{\delta}(x) = (\delta+1)(\delta+2)x^2 - (\delta+1)(2\delta+1)x + \delta^2, \ldots \end{aligned} \tag{12}$$

After applying the substitution $x = e^{-t}$ to Eq. (8) and Laplace transform, we have:

$$W_n^{\delta}(s) = \frac{\prod_{i=1}^{n}(s - i\delta)}{\prod_{i=0}^{n}(s+i)} = \frac{(s-\delta)(s-2\delta)\cdots(s-n\delta)}{s(s+1)(s+2)\cdots(s+n)} \tag{13}$$

III. Quasi-Almost orthogonal polynomials

The following definition of quasi orthogonality for polynomial set $P_n(x)$ can be found in papers [4, 5]:

$$\int_a^b P_n^k(x)P_m^k(x)w(x)dx = \begin{cases} = 0, & 0 \le m \le n-k-1 \\ \neq 0, & n \ge k+1 \end{cases} \tag{14}$$

where k represents the order of quasi orthogonality, a and b - quasi orthogonality interval and $w(x)$ - the weight function.

A large number of papers consider quasi orthogonal polynomials in classical manner, analogous to the classical orthogonal polynomials. One approach [4] is based on known theorem that every polynomial $R_n(x)$ that can be represent as linear combination of classical orthogonal polynomials given by:

$$R_n(x) = P_n(x) + c_1P_{n-1}(x) + \cdots + c_kP_{n-k}(x) \tag{15}$$

is itself quasi orthogonal with order k of quasi orthogonality. Coefficients c_i can be dependent on n. The second approach [4] is based on the fact that every orthogonal polynomial with weight $w(x)$ is quasi orthogonal with some other weight. For example, in the case of Legendre polynomials orthogonal over interval (-1, 1) with weight $w(x)=1$, we can introduce different weight: $w(x)=x^k$, and obtain quasi orthogonal, Legendre type, polynomials over same interval. Many papers also analyze in details configuration of the zeroes in quasi orthogonal polynomials [5, 12].

If we apply definition of quasi orthogonality on almost orthogonal polynomials given with Eq. (8), we obtain quasi-almost orthogonal Legendre type, polynomials over interval (0, 1) with weight function $w(x)=1$:

$$P_n^{k,\delta}(x)=\sum_{i=0}^{n} A_{n,i}^{k,\delta} x^i \tag{16}$$

where:

$$A_{n,i}^{k,\delta}=(-1)^{n+i+k}\frac{\prod_{j=1}^{n-k}(i+j\delta)}{i!(n-i)!} \tag{17}$$

Starting first order (k=1) quasi-almost orthogonal polynomials of this sequence are:

$$\begin{aligned}
&P_1^{1,\delta}(x)=-x+1,\\
&P_2^{1,\delta}(x)=-\frac{(\delta+2)}{2}x^2+(\delta+1)x-\frac{\delta}{2},\\
&P_3^{1,\delta}(x)=-\frac{(\delta+3)(2\delta+3)}{6}x^3+\\
&+(\delta+1)(\delta+2)x^2-\frac{(\delta+1)(2\delta+1)}{2}x+\frac{\delta^2}{3},\cdots
\end{aligned} \tag{18}$$

After applying the substitution $x=e^{-t}$ to Eq. (18) and Laplace transform, we have:

$$W_n^{1,\delta}(s)=\frac{\prod_{i=1}^{n-1}(s-i\delta)}{\prod_{i=0}^{n}(s+i)}=\frac{(s-\delta)(s-2\delta)\cdots(s-(n-1)\delta)}{s(s+1)(s+2)\cdots(s+n)} \tag{19}$$

Transfer function given in form of Eq. (19) is very suitable for practical design of quasi-almost orthogonal filters.

IV. Quasi-Almost Orthogonal Filters

First order (k=1) quasi-almost orthogonal polynomials can be generated by a filter whose analogue scheme is given in Fig. 1. This filter has been directly designed using relation (19). Signals labelled as $\varphi_i^{1,\delta}(t)$ in Fig. 1 represent the sequence of exponential quasi-almost orthogonal functions.

Obtained filter analogue scheme given in Fig. 1 is very simple and suitable for practical realization. Simulation in MATLAB, based on Fig. 1, was performed and few quasi-almost orthogonal signals (chosen value for δ was 1.02) are shown in Fig. 2. In this simulation we have applied input step signal with amplitude of 2V.

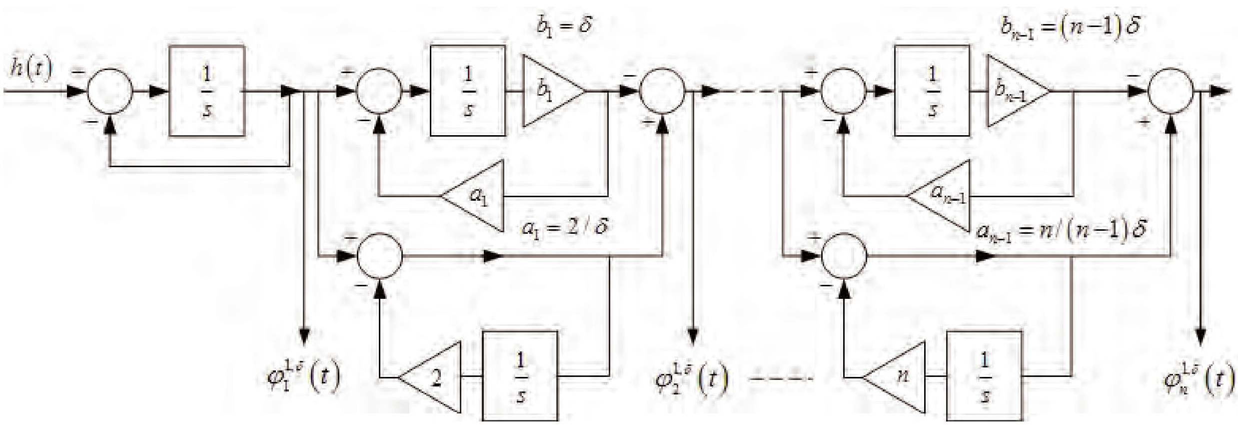

Fig. 1. Analogue scheme for the first order quasi-almost orthogonal filter adjusted for practical realization

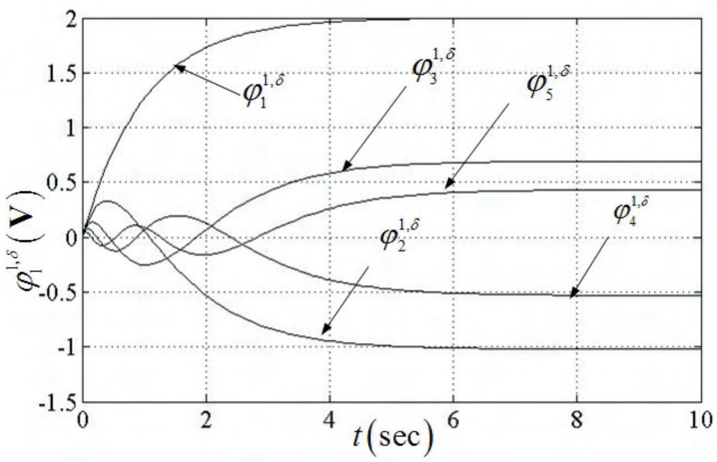

Fig. 2. First order quasi-almost orthogonal filter signals simulated in MATLAB

V. CONCLUSION

In this paper we have defined almost orthogonal polynomials, as well as quasi-almost orthogonal polynomials of Legendre type. Main relations valid for those polynomials are also given. Using these relations, we proposed a method for designing quasi-almost orthogonal filters of arbitrary order. Designed filter can be successfully used for modeling, identification, simulation, and analysis of different dynamical systems as well as for designing adaptive systems.

ACKNOWLEDGEMENT

This paper was supported in part by the Serbian Ministry of Science and Technology within the projects II 43007, II 44006 and TR 35005.

REFERENCES

[1] M. Riesz, "Sur le Probleme des Moments", Troisième Note, Arkiv för Matematik, Astronomi och Fysik, vol. 17(16), pp. 1-52, 1923.

[2] T.S. Chihara, "On Quasi-orthogonal Polynomials", Proceedings of the American Mathematical Society, vol. 8, pp. 765-767, 1957.

[3] S. Dehesa, F. Marcellan and A. Ronveaux, "On Orthogonal Polynomials with Perturbed Recurrence Relations", Journal of Computational and Applied Mathematics, vol. 30, pp. 203-212, 1990.

[4] C. Brezinski, K.A. Driver and M. Redivo-Zaglia, "Quasi-orthogonality with Applications to Some Families of Classical Orthogonal Polynomials", Applied Numerical Mathematic, vol. 48, pp. 157-168, 2004.

[5] M. Alfaro and L. Moral, "Quasi-orthogonality on the Unit Circle and Semi-classical Forms", Portugaliae Mathematica vol. 51, pp. 47-62, 1991.

[6] M. Milojković, S. Nikolić, B. Danković, D. Antić and Z. Jovanović, "Modelling of Dynamical Systems Based on Almost Orthogonal Polynomials", Mathematical and Computer Modelling of Dynamical Systems, vol. 16, no. 2, pp. 133-144, 2010.

[7] B. Danković, S. Nikolić, M. Milojković and Z. Jovanović, "A Class of Almost Orthogonal Filters", Journal of Circuits, Systems, and Computers, vol. 18, no. 5, pp. 923-931, 2009.

[8] S. Nikolić, M. Milojković, D. Antić, B. Danković, Z. Jovanović and S. Perić, "Almost Orthogonal and Quasi-orthogonal Filters", TEHNIKA-Elektrotehnika, vol. 59, no. 5, pp. 1-6, 2010.

[9] B. Danković, D. Antić, Z. Jovanović, S. Nikolić and M. Milojković, "Systems Modeling Based on Legendre Polynomials", SACI 2009, Timisoara, Romania, May 28.-29., pp. 241-246, 2009.

[10] S. Nikolić, D. Antić, B. Danković, M. Milojković, Z. Jovanović and S. Perić, "Orthogonal Functions Applied in Antenna Positioning", Advances in Electrical and Computer Engineering, vol. 10, no. 4, pp. 35-42, 2010.

[11] D. Antić, B. Danković, Z. Jovanović and S. Perić, "Legendre Orthogonal Functional Network Applied in Modeling of Dynamical Systems", ICEST 2010, Ohrid, Macedonia, June 23.-26., vol. 1, pp. 449-452, 2010.

[12] A. Bultheel, P. González-Vera, E. Hendriksen and O. Njastad, "Separation of Zeroes of Para-orthogonal Rational Functions", Acta Mathematica Hungarica, vol. 96, pp. 169-186, 2002.