A Parametric Identification Approach Based on the Final Value Theorem of the Laplace Transform

Milica B. Naumović[1] and Lazar Popović[2]

Abstract **–The paper describes the application of the finite value theorem of the Laplace transformin the identification procedure of a continuous-time parametric model. The presented identification approach involves only a number of integrating processes and is suitable for easy automation. Two examples are used to demonstrate the merits of the proposed identification algorithm.**

Keywords– **Identification, Step response, System modelling, Parameter estimation, Numerical simulation.**

I. INTRODUCTION

An important step in process control is the identification of a "suitable" model of a continuous-time system from real observations [1]. What suitable means depends primarily on the concrete application one has in mind.Thus, it is necessary to select an appropriate level of model complexity depending on the purpose of system identification.Moreover, the acceptance of models should be guided by "usefulness" rather than by "truth".

Many identification methods discuss the parameter estimation problems both of continuous-time and discrete-time system models.A detailed overview of such methods is given in Ljung [2]. Least-squares, step response, and frequency response methods are representative as deterministic off-line identification approaches.

The system identification techniques based on the continuous-time model were initiated in the middle of the last century [3], but, for some time, were overshadowed by the overwhelming developments in discrete-time methods. This was mainly due to the "go completely digitally" trend that was the result of the paralleldevelopment in digital computers.

This paper studies a deterministic off-line identification method ofthe rational transfer function which can be performed by using the data of a constant steady-state output step response. Such identification methodology, known as transient response analysis, is simple to apply and understand, and often providesonly information good enough for the estimates of the input-output gain, the dominant time constants, as well as the time delays. These properties make the methods suitable for the first-stage of the analysis to prepare for the other experiments in the system identification.

The paper is organized as follows.Section 2 introduces some preliminary facts before a simple identification method is presented. The properties of the identification procedure are summarized in Section III and illustrated by several simulation results.Finally some conclusions of the work are presented in Section IV.

[1]Milica B. Naumović is with the Faculty of Electronic Engineering, University of Niš, Aleksandra Medvedeva 14, 18000 Niš, Serbia, E-mail: milica.naumovic@elfak.ni.ac.rs

[2]Lazar Popović is with the Tagor Electronic, Tihomira Brankovića 21, 18000 Niš, Serbia, E-mail: lazar985@gmail.com

II. THE PARAMETRIC IDENTIFICATION METHOD

In this section, some assumptions are made for the plant to be identified, and an identification approach is considered.

A. The Plant to be Identified

A single-input single-output linear dynamical system can be described by a time-invariant linear differential equation

$$a_n c^{(n)}(t) + \cdots + a_1 c^{(1)}(t) + c(t) = b_m u^{(m)}(t) + \cdots + b_0 u(t) \qquad (1)$$

where $u(t)$ and $c(t)$ denote the input and output of the system, t represents time variable, and $c^{(i)}(t) \overset{\text{def}}{=} \mathrm{d}^i c(t) / \mathrm{d} t^i$, $u^{(i)}(t) \overset{\text{def}}{=} \mathrm{d}^i c(t) / \mathrm{d} t^i$. Coefficients a_i and b_i are the parameters of the system. Under the zero initial values, taking the Laplace transform to both sides of (1) and using the differential property,we can obtain

$$\left(a_n s^n + \cdots + a_1 s + 1\right) C(s) = \left(b_m s^m + \cdots + b_1 s + b_0\right) U(s) \qquad (2)$$

where $C(s)$ and $U(s)$ are the Laplace transforms of $c(t)$ and $u(t)$. Hence, we have the transfer function of the system

$$H(s) = \frac{C(s)}{U(s)} = \frac{b_0 + b_1 s + \cdots + b_m s^m}{1 + a_1 s + \cdots + a_n s^n}. \qquad (3)$$

Assume that $u(t)$ is a step function with the amplitude U_0.

Starting from the unit step response

$$\eta(t) = \frac{c(t)}{U_0}, \qquad (4)$$

by using the final value theorem of the Laplace transform

$$c(\infty) = \lim_{t \to \infty} c(t) = \lim_{s \to 0} sC(s), \qquad (5)$$

the coefficients of the model transfer function

$$\hat{H}(s) = \frac{\hat{b}_0 + \hat{b}_1 s + \cdots + \hat{b}_m s^m}{1 + \hat{a}_1 s + \cdots + \hat{a}_n s^n} \qquad (6)$$

can be determined.

B. The Algorithm of the Proposed Identification Method

Note that the method is based on the area determination and the graphical interpretation of the first step of the algorithm is shown in Fig. 1. According to the final value theoremofthe Laplace theorem, we have

$$\lim_{t\to\infty} \eta(t) = K_0 = \lim_{s\to 0} \hat{H}(s) = \hat{b}_0 , \qquad (7)$$

or

$$\hat{b}_0 = K_0 , \qquad (8)$$

where K_0 is indicated in Fig. 1.

It is suitable to define the integral $\eta_1(t)$ as follows [4]

$$\eta_1(t) = \int_0^t \left[K_0 - \eta(\tau)\right] \mathrm{d}\tau , \qquad (9)$$

and correspondingly the function

$$\hat{H}_1(s) = \frac{1}{s}\left[K_0 - \hat{H}(s)\right] = \frac{1}{s}\left[K_0 - \frac{\hat{b}_0 + \hat{b}_1 s + \cdots + \hat{b}_m s^m}{1 + \hat{a}_1 s + \cdots + \hat{a}_n s^n}\right]$$
$$= \frac{1}{s} \cdot \frac{\left(K_0 - \hat{b}_0\right) + \left(K_0 \hat{a}_1 - \hat{b}_1\right)s + \left(K_0 \hat{a}_2 - \hat{b}_2\right)s^2 \cdots + K_0 \hat{a}_n s^n}{1 + \hat{a}_1 s + \cdots + \hat{a}_n s^n} . \qquad (10)$$

The value

$$K_1 = \int_0^\infty \left[K_0 - \eta(\tau)\right] \mathrm{d}\tau \qquad (11)$$

can be calculated as it is indicated in Fig. 1. Than, the final value theorem of theLAPLACE transform gives

$$\lim_{t\to\infty} \eta_1(t) = K_1 = \lim_{s\to 0} \hat{H}_1(s) = K_0 \hat{a}_1 - \hat{b}_1 . \qquad (12)$$

The next step of integrating leads to the function $\hat{H}_2(s)$, the integral $\eta_2(t)$ and its limit value K_2, as follows:

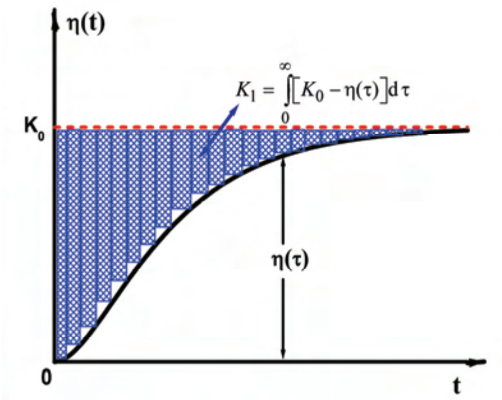

Fig. 1. Evaluation of the unit step response

$$\hat{H}_2(s) = \frac{1}{s}\left[K_1 - \hat{H}_1(s)\right]$$
$$= \frac{1}{s} \cdot \frac{\left(K_1 - K_0 \hat{a}_1 + \hat{b}_1\right) + \left(K_1 \hat{a}_1 - K_0 \hat{a}_2 + \hat{b}_2\right)s + \cdots}{1 + \hat{a}_1 s + \cdots + \hat{a}_n s^n} , \qquad (13)$$

$$\eta_2(t) = \int_0^t \left[K_1 - \eta_1(\tau)\right] \mathrm{d}\tau , \qquad (14)$$

and

$$\lim_{t\to\infty} \eta_2(t) = K_2 = \lim_{s\to 0} \hat{H}_2(s) = K_1 \hat{a}_1 - K_0 \hat{a}_2 + \hat{b}_2 . \qquad (15)$$

Continuing with the integration, a system of linear equations is obtained with a general formula as

$$(-1)^r \hat{b}_r + K_{r-1}\hat{a}_1 - K_{r-2}\hat{a}_2 + \cdots + (-1)^{r-1} K_0 \hat{a}_r = K_r$$
$$r = 0, 1, \ldots, n . \qquad (16)$$

Table I contains a summary of the identification procedure in the case when all b_j parameters except b_0 in (3) are equal to zero.

TABLE I
ALGORITHM FOR COMPUTING THE PARAMETERS OF THE TRANSFER FUNCTION IN (3)

$b_i = 0$, $i = 1, 2, \ldots, m$; $b_0 \neq 0$
$\hat{b}_0 = K_0$
$\hat{a}_1 = \frac{K_1}{K_0}$
$\hat{a}_2 = \frac{K_1}{K_0}\hat{a}_1 - \frac{K_2}{K_0}$
$\hat{a}_3 = \frac{K_1}{K_0}\hat{a}_2 - \frac{K_2}{K_0}\hat{a}_1 + \frac{K_3}{K_0}$
$\vdots$
$\hat{a}_n = \frac{K_1}{K_0}\hat{a}_{n-1} - \frac{K_2}{K_0}\hat{a}_{n-2} + \cdots + (-1)^{n+1}\frac{K_n}{K_0}$

Fig. 2 presents a simple block-diagram for performing all calculations in the described identification procedure. The measured data, obtained experimmentally from a real-time set-up, can be used inthe MATLAB®Simulink environment to identify unknown system parameters.It is obvious that the considered identification approach is very simple and can be realized with the minimalcomputational effort.

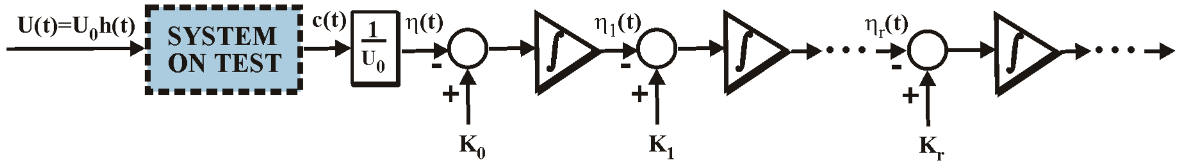

Fig.2. The schematic diagram of the considered identification procedure

III. Simulation Examples

In this paper, the described method will be illustrated by two examples related to the fourth-order linear objects without finite zeros.

A. Example 1.

Consider the following fourth-order linear system given by

$$H(s) = \frac{0.2}{0.2s^4 + 0.8s^3 + 2s^2 + 1.2s + 1}. \qquad (17)$$

In simulation, the input is taken as a step signal with the amplitude equal to 10. The important point is that the duration of the simulation should be sufficient to ensure that the input signal be able to excite the slowest system mode. The estimated parameters for a fourth-order model are shown in Fig. 3. It can be seen that after 25 seconds the estimated parameters converge to the true values.

B. Example 2.

Fig. 4 visualizes one of the variety of configurations to be obtaind with the ECP Model 210 Rectilinear Plant by using springs of varying stiffness [5]. A drive motor provides actuation to the system via the first mass, and position measurements $x_i(t)$, $i = 1,2$ are taken by quadrature encoders.

The equations for the considered mass-spring system may be found using Newton's laws to write force balance equations

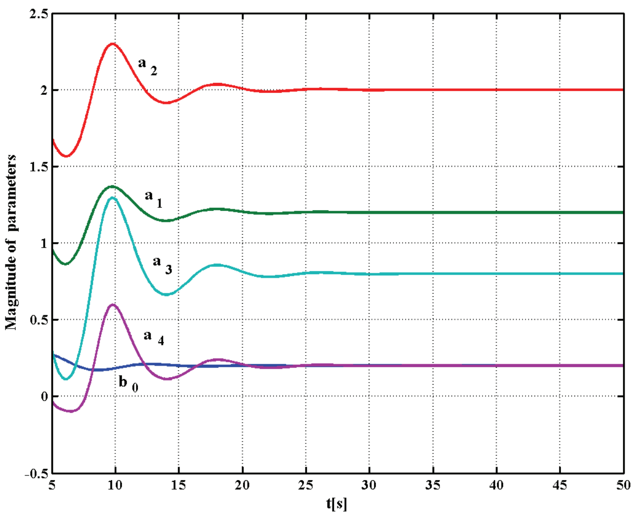

Fig.3. Estimated model parameters in Example 1

in matrix notation as

$$\mathbf{m}\ddot{\mathbf{x}}(t) + \mathbf{c}\dot{\mathbf{x}}(t) + \mathbf{k}\mathbf{x}(t) = \mathbf{F}(t), \qquad (18)$$

where

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},\ \mathbf{m} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix},\ \mathbf{c} = \begin{bmatrix} 0 \\ c_2 \end{bmatrix},$$

$$\mathbf{k} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}, \text{ and } \mathbf{F}(t) = \begin{bmatrix} f(t) \\ 0 \end{bmatrix}. \qquad (19)$$

The parameters of this set-up can be found in the literature[5], as follows: $m_1 = 1.7\,\text{kg}$, $m_2 = 1.2\,\text{kg}$, $k_1 = k_2 = 800\,\text{N/m}$, $k_3 = 450\,\text{N/m}$, $c_2 = 9\,\text{Ns/m}$. The above motion equation results in the transfer function

$$H(s) = \frac{X_2(s)}{F(s)} = \frac{392.2}{s^4 + 7.5s^3 + 1983s^2 + 7059s + 666700}, \qquad (20)$$

which can be equivalently rewritten in such a way as in (1) with:

$$b_0 = 5.8827 \cdot 10^{-4},\ a_1 = 0.0106,\ a_2 = 0.003,$$
$$a_3 = 1.1249 \cdot 10^{-5},\ a_4 = 1.4999 \cdot 10^{-6}. \qquad (21)$$

It should be noted that all a_i - parameters have low values, and moreover are of different order of magnitude, which is the known characteristic of the electro-mechanical systems.

For the purpose of the fourth-order model parameter estimation, the algorithm described in the previous section was applied. Table II presents the estimated values of the parameters obtained at some different conditions of experimentation. The results illustrate some of the fundamental problems of system identificationrelated to the experiment duration and the accuracy of data presentation. Thus, the electro-mechanical plant given in Fig. 4 can be

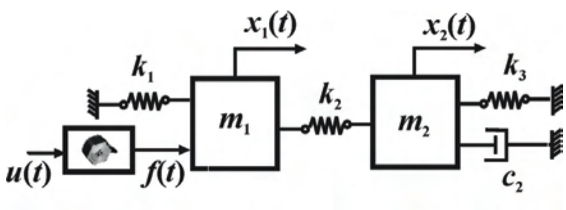

Fig.4. The scheme of the electro-mechanical plant

TABLE II
ESTIMATED PARAMETERS IN EXAMPLE 2

Estimated parameters	**Working formats in MATLAB® Simulink environment**		
	format long		**format short**
	Measured time $t = 20\text{s}$	**Measured time** $t = 10\text{s}$	$t = 20\text{s}$
$\hat{b}_0$	0.00058827058647	0.00058827056152	0.0006
$\hat{a}_1$	0.01058797060125	0.01058754730445	0.0106
$\hat{a}_2$	0.00297435128462	0.00297645908868	0.0032
$\hat{a}_3$	0.00001124942294	0.00000424347456	-0.0013
$\hat{a}_4$	0.00000149999767	0.00001896499007	0.0065
Efficiency	very good	poor	very poor

adequately represented by the model obtained after the identification procedure lasting 20 seconds and retaining the larger number of decimal places corresponding to the MATLAB® data presentation in long format.

IV. CONCLUSION

System identification is a well-established field. However, the search for the simple procedures of identification is still a special scientific challenge. This paper presents an identification algorithm implemented in the MATLAB®-Simulink environment based on the well-known final theorem of the Laplace transform. The properties of the described identification method are illustrated by the simulation results.

At the present stage, some conclusions can be drawn from the above study. The method is more difficult to implement if the model to be identified is given in the form of the transfer function with the finite zeros. The quality of the estimation has not be analyzed in the case of the noise corrupted system step responses.

ACKNOWLEDGEMENT

This paper is supported in part by project Grant III44004 (2011-2014) financed by Ministry of Education and Science, Republic of Serbia.

REFERENCES

[1] R. Pintelon, and J. Schoukens, *SystemIdentification: A Frequency Response Approach*, New York: IEEE Press, 2001.
[2] L. Ljung, *System Identification: Theory for the User*,Englewood Cliffs,NJ: Prentice-Hall, 1999.
[3] G.P. Rao, and H. Unbehauen, "Identification of continuous-time systems", Control Theory and Applications, IEE Proc, vol. 153, no. 2, pp. 185-220, 2006.
[4] H. Rake, "Step response and frequency response methods", Automatica, vol. 16, no. 5, pp. 519–526, 1980.
[5] T. R. Parks, "ECP Instructor's Manual for Model 210/210a Rectilinear Control System", 1999. Available: http://maeweb.ucsd.edu/~ugcl/download/manuals/linear.pdf (accessed on June 3rd 2011).