A Model of Vehicle Routing Problem with Soft Time Windows and Variable Traveling Time

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***Abstract* – This article extends the generalized vehicle routing problem model by introducing variable traveling time. An overview is given for the different approaches to formulate a model and a set of criteria is defined to choose the proper one. Based on the analysis a model is chosen and modified to fit the requirements of the real life problems. Analysis of the most promising algorithm families for solving the model is presented.**

***Keywords* – Vehicle routing problem, Multiple time windows, Variable traveling times.**

I. Introduction

The vehicle routing problems (VRP) are widely spread combinatorial optimization problems. They appear in practice during the solution of different logistics problems in different areas of everyday life. In the basic formulation of the problem a set of identical vehicles are based in a central depot. Vehicles have to be routed optimally to supply customers with known demands. The vehicles have known capacity and traveling costs. Each customer is a part of only one route and is visited only once by the vehicle serving the route. VRP is a NP-hard problem [24], since it is generalization of the well-known TSP problem where the number of vehicles is only one. VRP and its variants is applicable in many areas of social and economic life like postal services, bank deliveries, air cargo transportation, school bus routing and many others. This problem is relevant to all logistic services since one of the goals of modern society is reduction of the consumption of unrecoverable sources of energy and diminishing the environmental pollution.

VRP was first described by Dantzig and Ramser [10]. The problem was a real-world application concerning the delivery of gasoline to service stations. They proposed mathematical programming formulation and an algorithm to get near-optimal solution of a problem instance with twelve service stations and four trucks. After the publication of their paper the research in this area flourished.

Based on the practical needs different formulations of VRP appeared. On Fig. 1 some of the major sub-problems with the relations between them are given. The descriptions of the sub-problems below are based on the following model: Let $G = (V, A)$ be a complete graph where $V = 0, \ldots, n$ is the set of all vertices and A be the set of all arcs. Vertices $i = 1, \ldots, n$ correspond to the customers, whereas vertex 0 corresponds to the depot. A nonnegative cost c_{ij} is associated to each arc $(i, j) \in A$ and is the travel cost spent to go from customer i to customer j. Generally loop arcs (i, i) are not allowed and this is imposed by defining $c_{ii} = \infty$. Each customer i has an associated demand $d_i > 0$ to be delivered. The depot has a fictitious demand $d_0 = 0$. A set of K identical vehicles each with capacity C start to service a customer from the depot. To assure that there is a feasible solution it is assumed that $d_i = C; \quad i = 1, \ldots, n$. Each vehicle may perform only one route and it is assumed that K is not smaller than K_{min} where K_{min} is the minimum number of vehicles needed to serve all the customers. This minimum number can be determined by solving the Bin Packing Problem (BPP) derived from the VRP model. BPP is a NP-hard problem but the problem instances with hundreds of elements can be effectively solved to optimality [23]. All VRP sub-problems have to find K simple circuits, each circuit corresponding to a route, with minimum cost. The cost of a route is defined as the sum of costs of all arcs part of the route. A route should visit the depot vertex and each customer vertex is a part of exactly one route. Each of the sub-problems defined below extends this definition by adding additional constraints.

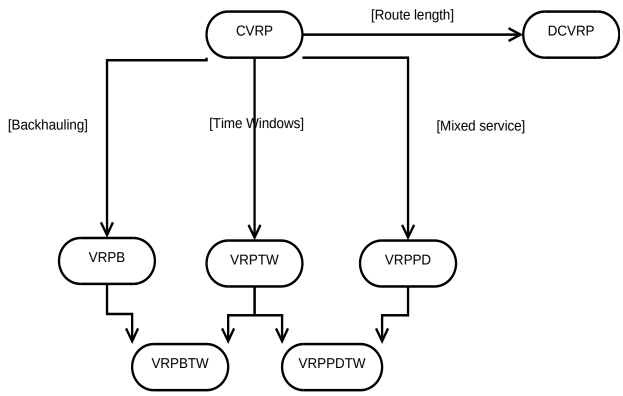

Figure 1. Major sub-problems of VRP and their relations [24]

- Capacitated VRP (CVRP) - to the basic model described above it is added that demands of the customers on the same route should not be greater than C;
- Distance-Constrained VRP (DCVRP) - based on the basic model where each route length or route duration should be less than a limit. There is no capacity constraint;
- VRP with Backhauls (VRPB) - extension of CVRP where customers set $V \setminus \{0\}$ is partitioned in two subsets. The first subset L contains l linehaul customers each requiring a given amount of products to be delivered. The second subset B contains n-l backhaul customers where a given quantity of inbound products must be picked up. Customers are numbered so that $L = \{1, \ldots, l\}$ and

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$B = \{l+1, \ldots, n\}$. In the VRPB there exists a precedence between linehaul and backhaul customers. If a route serves both types of customers linehaul customers should be served first. The total demand of linehaul and backhaul customers does not exceed, separately, the vehicle capacity *C*;

- VRP with Time Windows (VRPTW) - extension of CVRP where to each customer *i* a time interval $[a_i, b_i]$ called time window is associated. Customers can be served only in the associated time window. The vehicle stops at each customer on the route for s_i time instants. VRPTW is generalization of CVRP, when $a_i = 0$ and $b_i = +\infty$ for each $i \in V \setminus \{0\}$;
- VRP with Pickup and Delivery (VRPPD) - extension of CVRP where to each customer *i* two quantities d_i and p_i are associated representing the demand of homogeneous commodities to be delivered and picked up at customer *i*. Vehicles may pickup commodities from an origin customer *i* and deliver them to a destination customer *j*. The current load of the vehicle along the route must be nonnegative and may never exceed the vehicle capacity *C*. All customers part of an origin-destination relation should be part of the same route. There is also a precedence as origin customers should be served before destination customers;
- VRP with Backhauls and Time Windows (VRPBTW) - a combination between VRPB and VRPTW;
- VRP with Pickup and Delivery and Time Windows (VRPPDTW) - a combination between VRPPD and VRPTW;

There are three main approaches that can be found in literature [22, 24] to formulate a mathematical programming model for the basic VRPs presented above. The first type of models are known as vehicle flow formulations. They use integer variables, associated with each arc or edge of the graph which counts the number of times the arc or edge is traversed by the vehicle. These models do not keep track of the commodities delivered at each customer. Further they can be subdivided to two-index vehicle flow formulations, in which the variables are indexed according to the start and end vertices only, and the three-index vehicle flow formulations, in which the variables have a third index which distinguishes one vehicle from another. These models are more frequently used for the basic versions of VRP. They are particularly suited for cases in which the cost of the solution can be expressed as the sum of the costs associated with the arcs, and when the most relevant constraints concern the direct transition between the customers within the route, so they can be effectively modelled by an appropriate definition of the arc set and of the arc costs. On the other hand, vehicle flow models cannot be used to handle many practical issues, e.g., when the cost of a solution depends on the overall vertex sequence or on the type of the vehicle assigned to a route. Moreover, the linear programming relaxation of vehicle flow models can be very weak when the additional operational constraints are tight.

The second family of models is based on the so-called commodity flow formulation. In this type of models, additional integer variables are associated with the arcs or edges and represent the flow of the commodities along the paths travelled by the vehicles. Vehicle flow models have an exponential number of constraints to enforce connectivity while commodity flow models impose this requirement by using a set of continuous variables representing the flow of one or more commodities between the depot and the customers. They were introduced by Garvin et al. [14] and later extended by Gavish and Graves [15, 16].

The models of the last type have an exponential number of binary variables, each associated with a different feasible circuit. The VRP is then formulated as a Set-Partitioning Problem (SPP) calling for the determination of a collection of circuits with a minimum cost, which serves each customer once and, possibly, satisfies additional constraints. For the first time these models were proposed by Balinski and Quandt [2]. A main advantage of this type of models is that it allows extremely general route costs, e.g., depending on the whole sequence of the arcs and on the vehicle type. Moreover, the additional side constraints need not take into account the restrictions concerning the feasibility of a single route. As a result, they often can be replaced by a compact set of inequalities. This produces a formulation whose linear programming relaxation is typically much tighter than that in the previous models. Note, however, that these models generally require dealing with a very large number of variables.

II. A Model of Vehicle Routing Problem with Soft Time Windows and Variable Traveling Time

The vehicle routing problem for which a model is presented is based on a real-world routing problem. It is generalization of VRPTW where multiple time windows are supported. The service takes place entirely in the city area. This means that the main city roads are overloaded during the rush hours and the travel duration for a road segment will vary significantly depending on the time of the day. The city infrastructure does not provide paid highways that can be used to construct more expensive but faster routes. The average number of customers to be served is 600. The number of vehicles is up to 40 and they are identical. The model is robust enough to allow future modifications such as addition of new constraints or changes in the objectives.

A model is introduced for a vehicle routing problem with soft time windows and variable traveling times. The model is based on the model presented in [19] and is chosen because of its simplicity and flexibility. Let $G = (V, E)$ be a complete directed graph with a vertex set $V = \{0, \ldots, n\}$ and an edge set $E = \{(i, j) \mid i, j \in V, i \neq j\}$ and $K = \{1, \ldots, k\}$ be the vehicle set. Vertex 0 is the depot and the other vertices are the customers to be served. Each customer *i*, each vehicle *k* and edge $(i, j) \in E$ is associated with:

- $g_i \geq 0$ - the demand of goods of customer *i*;

- $p_i(t)$ - time window cost function of the start time *t* of the service at customer *i*;
- $p_0(t)$ - time window cost function of the arrival time *t* at the depot;
- $C_k \geq 0$ - the capacity of vehicle *k*;
- $d_{ij} \geq 0$ - the distance between vertices *i* and *j*;
- $q_{ij}(t)$ - traveling time function from *i* to *j*;

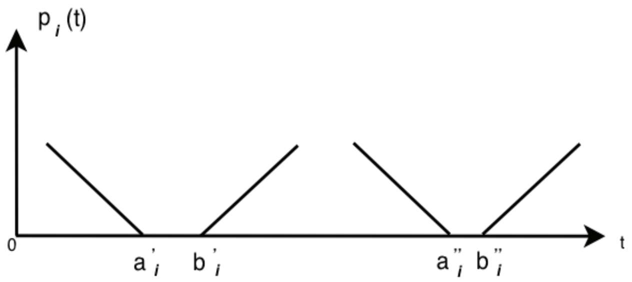

Figure 2. Cost function $p_i(t)$ for two time windows

Function $p_i(t)$ may be piecewise linear and look as given on Fig. 2. The actual slope may be chosen according to the preferences for the real problem instance and affects how 'soft' is a time window. The model can be transformed to hard time windows if $p_i(t)$is defined as follow:

$$p_i(t) = \begin{cases} 0 & t \text{ within the time window} \\ +\infty & t \text{ outside the time window} \end{cases}$$

Function $q_{ij}(t)$ is different from the described in [19]. In the proposed model the function will reflect the different traveling times for the same road depending on what time of the day the vehicle passes it. It takes the current time and returns the estimated travel duration between customers *i* and *j*.

Let σ_k be the route traveled by vehicle *k* and $\sigma_k(h)$ be the *h*th customer in route σ_k. By n_k the number of nodes on route σ_k is given. For convenience $\sigma_k(0) = \sigma_k(n_k + 1) = 0$ for all *k* (i.e. every route will start and finish at the depot). With σ the set of the routes followed by the vehicles will be denoted

$$\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_k)$$

Additionally let us introduce the following notations:

- s_i - the start time of service at customer *i*;
- s_k^a - the arrival time of vehicle *k* at the depot;
- $s = (s_1, s_2, \ldots, s_n, s_1^a, s_2^a, \ldots, s_n^a)$;

Let us introduce also binary variables $y_{ik}(\sigma) \in \{0,1\}$ for $i \in V \setminus \{0\}$ and $k \in K$ by

$$y_{ik}(\sigma) = 1 \Leftrightarrow i = \sigma_k(h) \quad (1)$$

Equation (1) holds for exactly one $h \in \{1, 2, \ldots, n_k\}$. This means that $y_{ik}(\sigma) = 1$ holds if and only if vehicle *k* visits customer *i*. We can express the total distance traveled by all vehicles as $d(\sigma)$, the total time window cost for all customers $p(s)$, the total traveling time cost $q(\sigma, t)$ and the total excess amount $g(\sigma)$ as follows:

$$d(\sigma) = \sum_{k \in K} \sum_{h=0}^{n_k} d_{\sigma_k(h), \sigma_k(h+1)} \quad (2)$$

$$p(s) = \sum_{i \in V \setminus \{0\}} p_i(s_i) + \sum_{k \in K} p_0(s_k^a) \quad (3)$$

$$q(\sigma, t) = \sum_{k \in K} \sum_{h=0}^{n_k} q_{\sigma_k(h), \sigma_k(h+1)}(t) \quad (4)$$

$$g(\sigma) = \sum_{k \in K} \max \left\{ \sum_{i \in V \setminus \{0\}} g_i y_{ik}(\sigma) - C_k, 0 \right\} \quad (5)$$

The mathematical programming model will be:
Minimize

$$\mathrm{c}(\sigma, s, t) = d(\sigma) + \alpha p(s) + \beta q(\sigma, t) + \gamma g(\sigma) \quad (6)$$

subject to

$$\sum_{k \in K} y_{ik}(\sigma) = 1, \quad i \in V \setminus \{0\} \quad (7)$$

Constraint (7) requires that every customer $i \in V \setminus \{0\}$ must be served only once by exactly one vehicle. The proposed objective function is a weighted sum of $d(\sigma) + \alpha p(s) + \beta q(\sigma, t) + \gamma g(\sigma)$. The constants $\alpha \geq 0, \beta \geq 0$ and $\gamma \geq 0$ in (6) determine the relative importance of each component of the objective function. They are set in advance depending on the preferences for each specific problem instance.

From the problem description it is obvious that the best approach to solve it will be to use a heuristic algorithm. Current methods to solve VRP problems to optimality use Branch-and-Bound, Branch-and-Cut [24], Branch-and-Cut-and-Price [12] or Set-Covering-Based algorithms [1]. They are able to solve problem instances of up to about 100 customers with variable success rate [21]. For larger problem instances both academic research [21] and commercial organizations [18] concentrate on heuristics. Heuristics are also more flexible than the exact approaches [21] and will be easily adapted to changes in the problem model. Heuristic and metaheuristic approaches are intensively researched in the recent years [5–7]. VRP with time windows has wide range of applications and is well studied [4, 13, 17]. Mathematical models with multiple time windows are also present in literature [11, 20].

Many of the heuristics and metaheuristics for VRPTW show variable performance and are also very dependent on the quality of the initial solutions [8]. As an improvement a multistart local search heuristic approach is proposed. It consists of two phases with an optional post-optimization procedure. In the first phase sequential insertion heuristics is used to generate a set of initial solutions. In the second phase inter and intra-route cross-exchange is invoked to reduce the total distance of the solutions with the minimum number of routes. In [19] this model is solved using an algorithm based on iterated local search. It starts from an initial solution σ and searches for a better solution in the neighborhood $N(\sigma)$. Standard neighborhoods as 2-opt*, cross-exchange and Or-opt with slight modifications are used for neighborhoods $N(\sigma)$. Approaches using behaviorally inspired algorithms are also present in the literature. Bee colony algorithm is proposed in [17]. Ant colony metaheuristic is proposed for VRP with multiple time windows in [9].

The model proposed can be considered to be a multicriterial one if we treat functions $d(\sigma), p(s), q(\sigma, t)$ and $g(\sigma)$ as separate objectives. In this formulation it is not necessary to determine in advance the relative importance of the components in (6). VRPTW and its variants are intrinsically multicriterial in nature [3] since the different components in (6) are related to each other. For example, the minimization of the total traveling cost $q(\sigma, t)$ may cause the vehicle to miss a few or more time windows at some customers, which will increase the value of $p(s)$. By using the multicriterial optimization approach we can find different alternative solutions and the final solution to be left to decision makers depending on the context of the specific problem instance.

III. Conclusion

In order to be applicable to a wider range of real life problems, one of the main features of a model should be its flexibility. The model should be adapted easy to variations of the constraints or the formulation of the goals. The presented model covers these requirements and is a good starting point to apply different metaheuristic approaches. Metaheuristics are suitable because they may provide good trade-off between speed, quality of the solution and the size of the problem input. The directions for future work are research on interactive algorithms guided by expert, application of parallel algorithms to the model and development of multicriterial algorithms for this model.

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