Digital Bandpass IIR Filers with High Selectivity

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***Abstract* – This paper proposes an optimal third-degree polynomial, which approximates Kronecker's delta function with high precision. The polynomial is obtained by a new approximation method, called "method with compressed cosines". The method is based on Chebyshev's optimality norm. The polynomial is used for narrow bandpass IIR filter design. The filter's selectivity depends on the parameter *Q* without increasing the polynomial's order. With the proposed method an IIR filter with 5(6) multipliers, a very narrow passband and a high stopband attenuation can be designed.**

Keywords – **IIR digital filters, Frequency response, Polynomial approximation.**

I. INTRODUCTION

The task of filter synthesis is a mathematical problem of approximating ideal functions with rectangular shape. The transfer function of the filter results from the approximation. The aim is to obtain a mathematical relationship which has the lowest computational complexity and approximation error. In approximations with polynomials, this indicator is the degree of the polynomial.

This paper will show a method for digital filter design based on a polynomial approximation with "compressed cosines".

II. BACKGROUND

In some practical cases the passband filter is required to have a very narrow bandwidth. The ideal characteristic of a supernarrowband filter is Kronecker's delta function

$$\delta(x)=\begin{cases}1, & x=0.5\\ 0, & x\neq 0.5\end{cases};\ x\in[0,1]. \qquad (1)$$

This is a transfer function of a filter that has a pass bandwidth equal to zero, stopband gain equal to zero and an infinite steepness of its characteristic. It cannot be realized in practice. Hence, Kronecker's delta function needs to be approximated by another one, which can be realized. The approximation is carried out with a preset accuracy $\varepsilon>0$. The difference between the ideal function and the approximating polynomial defines the error function. The two most popular norms for the approximation are L_2 - weighted integral least-squares norm, and L_∞ - weighted Chebyshev's norm. In the literature different polynomial approximation methods are proposed. Fig.1 shows approximations with Hausdorff (Chebyshev) polynomial [1] using L_∞ norm, with $\mathrm{sinc}(.)$

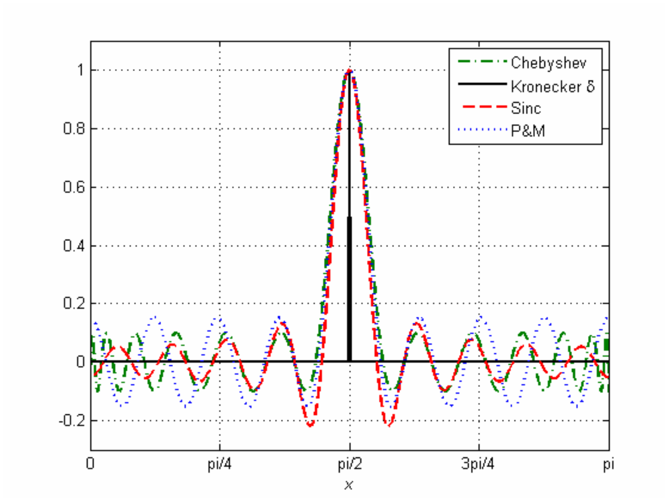

Fig. 1. Polynomial approximations of the Kronecker's delta

function using L_2 norm, method of Parks-McClellan [2] with trigonometric polynomial using L_∞ norm.

It is seen that a suitable trade-off between the flatness in the stopband and the bandwidth must be done. In all the criteria, the functions have the oscillations in the stopband. These oscillations are undesirable. The goal is to obtain a rectangular shape of the ideal function, that has maximally flat pass band and stop band, and narrowest possible bandwidth. In L_2 case the oscillations increase near the main lobe. This is due to the Gibbs' phenomenon [3]. In the approximations using L_∞ norm the oscillations are with equal amplitude. These approximations are known as optimal and equiripple.

The approximations with rational functions [4, 5], have better properties than the polynomials approximations. The most popular are Chebyshev, Butterworth and Cauer.

In [5, 6] a polynomial approximation method with compressed cosines is proposed. With this method a third degree polynomial with significantly better properties than the other polynomials approximations is derived. The approximation accuracy is close to the approximations with rational functions. The polynomial has the form

$$P_3=\sum_{k=1}^{4} b_k\cos\left[(k-1)\varphi\right], \qquad (2)$$

with coefficients:

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$$b_1 = 0.5 - \varepsilon;\ b_2 = b_4 = 0;\ b_3 = -0.5\,. \tag{3}$$

The function

$$\varphi = -\pi - 2\mathrm{arctg}\left[Q\left(\frac{2f - 2f_0}{f_d} - \frac{f_d}{2f - 2f_0}\right)\right], \tag{4}$$

is the phase response off the allpass lattice filter with quality factor Q, $f \in [0, f_d/2]$ is frequency, f_d is the sampling

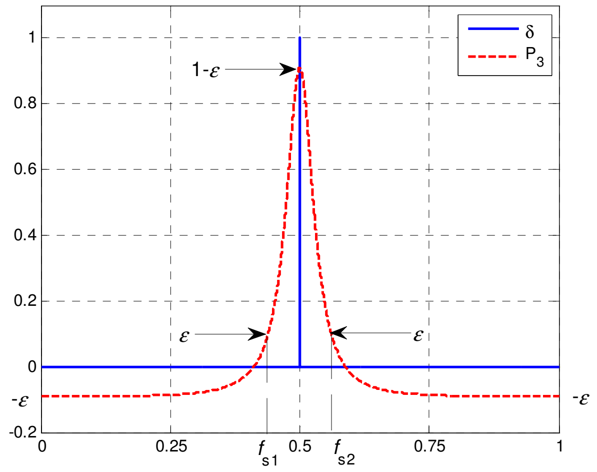

Fig. 2. Approximation of the Kronecker's delta by a third-degree optimal polynomial

frequency, f_0 is the middle frequency of the passband. Fig. 2 shows an optimal approximation of Kronecker's delta function by an optimal 3rd degree polynomial . f_{s1} and f_{s2} are the two normed stopband frequencies. Their difference defines the bandwidth Δf_{stop}. The passband is defined by Δf_{pass} - the bandwidth at level -3dB. The approximation error ε determines the stopband attenuation DS, and the quality factor Q the bandwidth Δf_{stop}. The filter's coefficients are obtained by those of the polynomial:

$$h_k = b_4/2, b_3/2, b_2/2, b_1, b_2/2, b_3/2, b_4/2\,. \tag{5}$$

The filter's transfer function has the form

$$H = -0.25 + (0.5 - \varepsilon)\exp(-j\varphi) - 0.25\exp(-j2\varphi)\,. \tag{6}$$

III. Design Example

The realization will be demonstrated with an example of a bandpass digital IIR filter design with the following specification: middle frequency in the pass band $f_0 = 800$ Hz; $\Delta f_{pass} = 100$ Hz; sampling rate $f_d = 8000$ Hz; degree of the polynomial $m = 3$; attenuation in the stopband $DS \geq 20$ dB.

A normalization of the frequencies is done as

$$\Delta f_n = 2\Delta f_{pass}/f_d = 0.025;\ f_{0n} = 2f_0/f_d = 0.2\,. \tag{7}$$

The optimal approximation error is determined as

$$\varepsilon = \frac{1}{1 + 10^{DS/20}} = 0.0909\,. \tag{8}$$

When the normalized bandwidth Δf_n is set, the Q-factor can be determined approximately by the following five equations:

$$Q = f_{0n}\sum_{k=1}^{11} a_k \Delta f_n^{\,11-k};\ \Delta f_n \in [0.1, 0.0127); \tag{9}$$

a_1=5.6075e14	a_2=-3.348e14	a_3=8.7823e13	a_4=-1.3321e13
a_5=1.2944e12	a_6=8.4311e10	a_7=3.7445e9	a_8=-1.1304e8
a_9=2.2671e6	a_{10}=-2.8705e4	a_{11}=1885.71	

$$Q = f_{0n}\sum_{k=1}^{11} a_k \Delta f_n^{\,11-k};\ \Delta f_n \in [0.0127, 1.272\mathrm{e}-3); \tag{10}$$

a_1=1.2838e25	a_2=-9.0581e23	a_3=2.79655e22	a_4=-4.9689e20
a_5=5.6239e18	a_6=-4.2396e16	a_7=2.1628e14	a_8=-7.435e11
a_9=1.6817e9	a_{10}=-2.3764e6	a_{11}=206.1499	

$$Q = f_{0n}\sum_{k=1}^{11} a_k \Delta f_n^{\,11-k};\ \Delta f_n \in [1.272\mathrm{e}-3, 3.14\mathrm{e}-4); \tag{11}$$

a_1=9.934e34	a_2=-7.838e32	a_3=2.7414e30	a_4=-5.6059e27
a_5=7.4403e24	a_6=-6.7267e21	a_7=4.2269e18	a_8=-1.8471e15
a_9=5,5072e11	a_{10}=-1.0692e8	a_{11}=1.2226e4	

$$Q = f_{0n}\sum_{k=1}^{11} a_k \Delta f_n^{\,11-k};\ \Delta f_n \in [3.14\mathrm{e}-4, 5.965\mathrm{e}-5); \tag{12}$$

a_1=-1.6808e41	a_2=2.5067e38	a_3=-1.4578e35	a_4=3.5739e31
a_5=1.3647e27	a_6=-3.3608e24	a_7=1.016e21	a_8=-1.6545e17
a_9=1.6554e13	a_{10}=-1.0265e9	a_{11}=36979.1	

$$Q = f_{0n}\sum_{k=1}^{11} a_k \Delta f_n^{\,11-k};\ \Delta f_n \in [5.965\mathrm{e}-5, 5.5\mathrm{e}-6)\,. \tag{13}$$

a_1=-4.7623e49	a_2=1.2294e46	a_3=-1.1141e42	a_4=2.0403e37
a_5=3.8601e33	a_6=-3.5797e29	a_7=1.5185e25	a_8=-3.7635e20
a_9=5.697e15	a_{10}=-5.176e10	a_{11}=263267.1	

By substituting the defined in (7) $\Delta f_n = 0.025$ into (9), $Q = 5.0714$ is obtained. This allows for defining the transfer function of the allpass lattice filter. The coefficients of the denominator of the transfer function are determined by those of the denominator of the Butterworth bandpass filter of first order with bandwidth

$$\Delta f_{Butt} = f_0/Q = 157.7469 \text{ Hz}. \tag{14}$$

The coefficients in the numerator are the same as in the denominator, but in reverse order.

$$H_{AP}(z)=\frac{c_1+c_2z^{-1}+z^{-2}}{1+c_2z^{-1}+c_1z^{-2}};\ z=\exp(-j\omega);$$

$$c_1=0.8832;\ c_2=-1.5265. \tag{15}$$

Then

$$\exp(-j\varphi)=H_{AP}(z)=\frac{0.8832-1.5265z^{-1}+z^{-2}}{1-1.5265z^{-1}+0.8832z^{-2}}. \tag{16}$$

In Fig.3, the diagram of the designed filter is shown. A criterion for comparing the selectivity of the digital filters is the number of multipliers with which they are realized.

The coefficients having the same value are realized with one multiplier to reduce the power consumption. As it is known, the allpass lattice filters are realized with 4 multipliers. Therefore, the total number of multipliers in the diagram is 11. This scheme can be realized with only six multiplier as both allpass filters are the same and $h_1=h_3=-0.25$. If the filter's coefficients are multiplied by 4, then $h_1=h_3=-1$. Then the filter will be realized with 5 multipliers, as in digital signal processing the change of the sign with the operation $x=-x$ is performed. In this case the filter will amplify the signal four times (12dB).

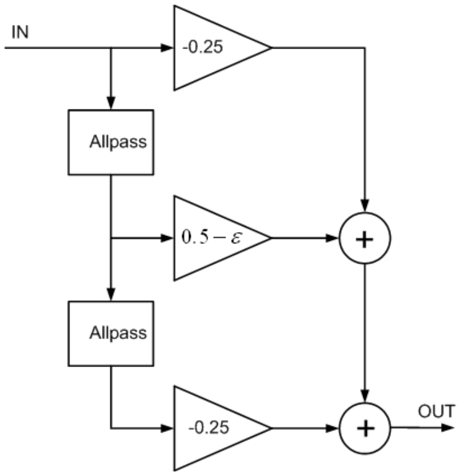

Fig. 3. Functional diagram of the filter

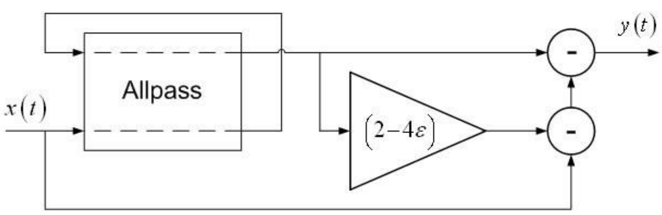

Fig. 4. Functional diagram of the filter with 5 multipliers

The scheme shows that all signals are summed. Therefore in the design the sequence of the operations is irrelevant, in accordance with the commutative law. This allows the input signal to pass twice through one allpass lattice filter. The bandpass filter is implemented with the scheme in Fig.4

Fig. 5 shows the magnitude response in dB $(10\lg x)$. Fig. 6 shows an output response of a computer simulation of the filter with 5 multipliers. The filter input is fed with a discretized at 8000Hz linear chirp signal with frequency sweep from 1 to 4000Hz, amplitude of ± 0.25V and duration of 10 seconds. It is seen that the filter's output response corresponds to the input specification.

This implementation requires frame signal processing with buffers. The concatenation of two neighbor fragments is treated with "overlap" to remove the uncertainties, which result from the filtration at the beginning of each fragment.

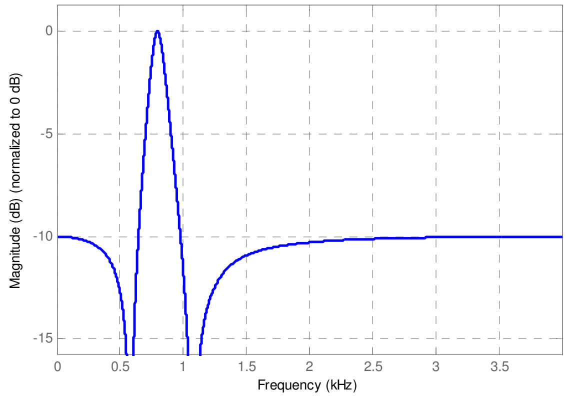

Fig. 5. Magnitude response

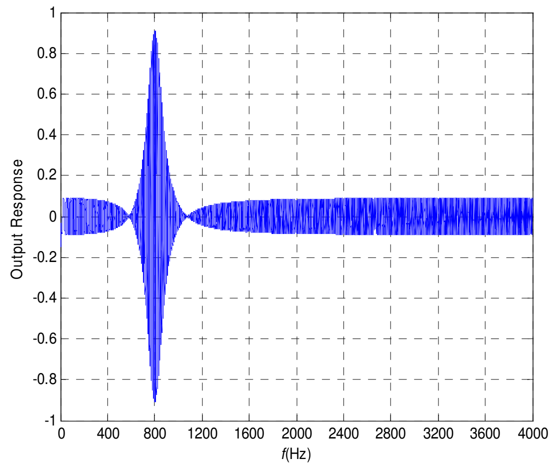

Fig. 6. Computer simulation – output response

IV. DISCUSSION

An advantage of the method of compressed cosines is that the approximation is carried out with third-degree polynomial. The polynomial's coefficients are calculated easily. To obtain a high selectivity, it is not necessary to increase the degree of the polynomial, as in other polynomial approximations, but to use an allpass lattice filter with high Q-factor. The bandwidth of the stopband Δf_{stop} is a result of the approximation. It can not be defined in the input specification. With a third-degree polynomial, a filter with an arbitrary bandwidth Δf_{pass} and a stopband attenuation DS can be realized. For example, Fig. 7 shows the magnitude response of a filter with a passband of 1Hz and a stopband attenuation of 60dB.

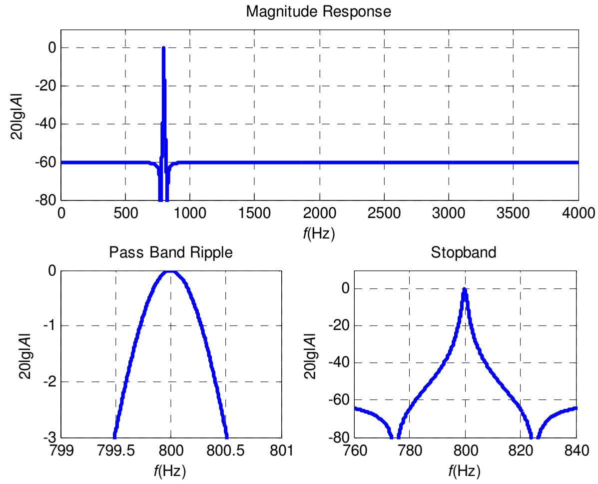

Fig.7. Bandpass filter - magnitude response

The most commonly used IIR digital filters are those of Butterworth, Chebyshev and Cauer. A criterion for comparing the selectivity is the number of multipliers.

1. The scheme of the digital filter of Fig. 4 is realized always with 5 multipliers, regardless of the filter's specification. This comes at the expense of using a larger volume of the memory, which is not a serious disadvantage. The filters of Butterworth, Chebysev and Cauer of first order are implemented by 4 multipliers. Due to the low order, their magnitude responses are identical. Fig.8 compares magnitude responses of bandpass filters with equal banwidth Δf_{pass}. The magnitude response of the filter using compressed cosines has better selectivity.

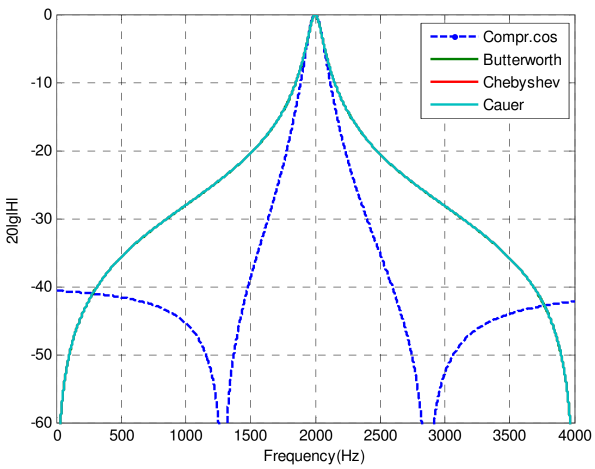

Fig. 8. Comparison of magnitude responses

2. Butterworth, Chebysev and Cauer filters of second order are implemented by 7-9 multipliers. Fig.9 shows a similar comparison. In this case the magnitude response of the filter using compressed cosines with 5 multipliers has a lower selectivity.

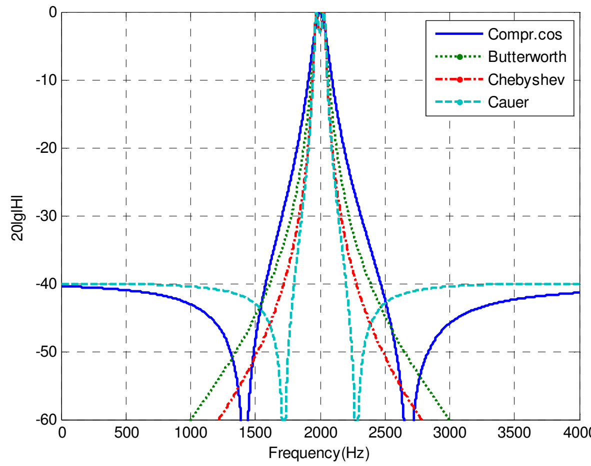

Fig. 9. Comparison of magnitude responses

V. Conclusion

The obtained results show that the selectivity of the filters with "compressed cosines" is determined by the steepness of the S-curve of the allpass filter's phase response (Q-factor). From (4) it is seen that it is the function arctg(.). To obtain a high selectivity it is necessary to use a function with a greater gradient, e.g. tanh(.). Unfortunately, an allpass filter with such a phase response has not been realized until now.

The proposed method may be a good alternative in several applications in IIR bandpass filter design.

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