Accuracy Improvement of Allpass-based Digital Hilbert Transformers

Kamelia Nikolova[1] and Georgi Stoyanov[1]

Abstract – **A design procedure to reduce the deviation of the phase from 90º of allpass-based digital Hilbert transformers is proposed. This is achieved by introducing the phase sensitivity minimization of each individual allpass section in the cascade realizations of the two branches of the structure used. The effectiveness of the proposed design is experimentally proven.**

Keywords – **digital filters, allpass filters, Hilbert transformers, sensitivity minimizations.**

I. INTRODUCTION

Hilbert transformers (HT) are very important building blocks in both, analog and digital signal processing. They are used in telecommunications for generation of analytic and single-sideband signals [1] [2] and in many other modulation and demodulation schemes (mainly for splitting the narrow-band signals to two (*I* and *Q*) components), in complex signals processing, in audio and video signal processing, and even in fields like mechanical vibration signal processing. Many approaches and methods of design of digital HTs have been developed in the last 50 years and most of them have been well systematized in [3]. The FIR based HTs are providing easily a linear phase response and unconditional stability but at the price of a very high transfer function (TF) order (say, several hundred), producing quite a high total delay and requiring higher power consumption. These disadvantages are eliminated in the IIR realizations, most often based on the usage of allpass structures. The theory of the allpass-based HTs is quite mature and several design methods using real or complex allpass structures have been summarized in [3]. Many new optimization-based methods for design of half-band filters and HTs have been proposed since then (including even frequency response masking technique [4]), but no specific methods for accuracy improvement have been reported. Meanwhile the practical importance of the HTs grew considerably with the extension of the frequency ranges and the growth of the proportion of the narrow-band signals, described as analytic, in telecommunications. The problem with the accuracy of the realization of the HTs is of paramount importance in many of these telecommunication applications, like in the maintenance of *I* and *Q* channels balance in a wide frequency range. When the HTs are realized using a fixed-point arithmetic (what is often the case in the portable and mobile communication equipment), the limited word-length may reduce considerably that accuracy and special measures have to be taken to prevent that. Higher accuracy could be achieved by designing the HTs with higher TF order, but the portability of the equipment is imposing another constraint – the power supply limitation. The main aim of this work is to try to improve the accuracy of the allpass-based HTs throughout minimization of their sensitivities. It will reduce the computational load and will permit shorter word-length and lower power consumption for given accuracy. The design procedures should be straightforward, without iterative and complicated optimization steps, in order to be easily used by practicing engineers and the structures have to be with the lowest possible TF order and complexity.

II. DESIGN PROCEDURE

An ideal Hilbert transformer (also known as a 90-degree phase shifter) is described in frequency domain as [5]

$$H_{HT}(e^{j\omega}) = \begin{cases} -j, & 0 \le \omega < \pi \\ j, & -\pi \le \omega < 0 \end{cases}. \qquad (1)$$

A way to synthesize an IIR Hilbert transformer (called also a complex half-band filter) is to start with an odd-order half-band filter with specifications F_p, F_s, δ_p and δ_s, interconnected by the relations [3]

$$\delta_s = \sin(\Delta\varphi_{\max}/2);\ \delta_p = 1 - \sqrt{1-\delta_s^2};\ F_p = 0.5 - F_s; \qquad (2)$$

and with a TF $G(z)$ that may be represented as a sum of two allpass TFs [3] [5]

$$G(z) = 0.5[A_1(z^2) + z^{-1}A_2(z^2)]. \qquad (3)$$

An "even-odd" decomposition (Fig. 1) and the substitution

$$H(z) = 2G(-jz) \qquad (4)$$

must be applied in order to obtain the real allpass TFs. Thus

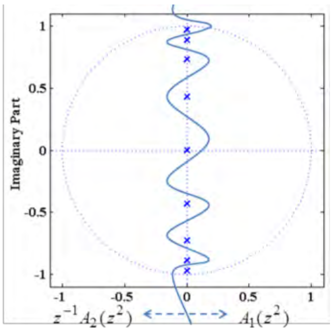


Fig. 1. "Even-odd" decomposition of the TF poles.

$$H_{HT}(z) = 2G(-jz) = [A_1(-z^2) + jz^{-1}A_2(-z^2)] \qquad (5)$$

represents the HT as a complex sum of two real allpass functions, whose realization (for real input signal $x(n)$) is given in Fig. 2. Details about the design are given in [3] [5].

---

[1] The authors are with the Faculty of Telecommunications at Technical University of Sofia, 8 Kl. Ohridski Blvd, Sofia 1000, Bulgaria; e-mails: ksi@tu-sofia.bg, stoyanov@ieee.org.

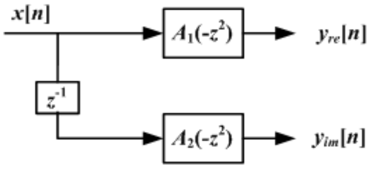
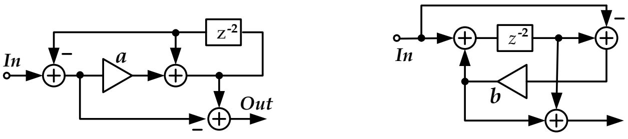


Fig. 2. HT realization.

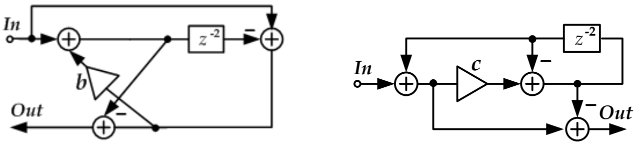
III. ALLPASS SECTIONS REALIZATIONS

The allpass TFs in Eq. (3) are having all their poles on the imaginary axes, while those in Eq. (5) are all on the real axes. In order to obtain higher accuracy in the 90º phase shifting in case of a limited word-length environment, the allpass TFs in Fig. 2 could be realized as cascades of special second-order allpass sections. It follows from Fig. 1 that if a cascade realization would be used, as the possible real pole positions are scattered all around the real axes, the allpass sections with low sensitivities for all these positions will be needed.

We have studied [6] all known (about 20) first order sections and it was found that several low-sensitivity sections for every real pole position could be found. We select to use the most typical four of them, namely the *ST*1 section, providing low-sensitivity for poles near z=1, *MH*1 and *SC*, having low sensitivity for poles near z=0 and *SV* section for poles near z=-1. The special sections are obtained from these real first order sections by changing the signs of the coefficients of the allpass TFs in Eq. (3) and by replacing z^{-1} by z^{-2} as it is shown in Fig. 3. We denote these new second order allpass sections as *MH*1-2, *ST*1-2, *SV*-2 and *SC*-2.



(*a*) *ST*1-2 (*b*) *MH*1-2



(*c*) *SC*-2 (*d*) *SV*-2

Fig. 3. Different special second-order allpass sections.

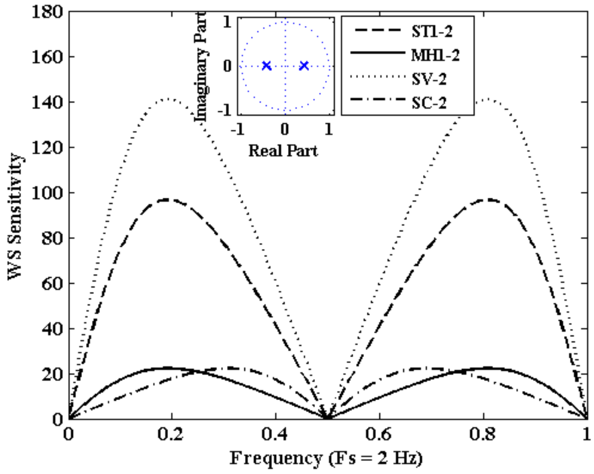
Their TFs are:

$$H_{ST1-2}(z)=\frac{-(1-a)+z^{-2}}{1-(1-a)z^{-2}};\quad H_{MH1-2}(z)=\frac{-b+z^{-2}}{1-bz^{-2}};\quad (6)$$

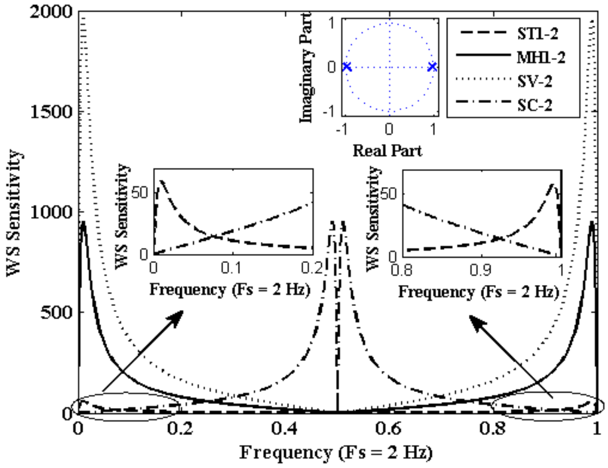
$$H_{SC-2}(z)=\frac{-b-z^{-2}}{1+bz^{-2}};\quad H_{SV-2}(z)=\frac{1-c+z^{-2}}{1+(1-c)z^{-2}}.\quad (7)$$

IV. ALLPASS SECTIONS SENSITIVITY INVESTIGATIONS

In Fig. 4 a, b the worst-case (WS) phase-response sensitivities of the above mentioned four special sections are given for realizations with two different TF pole positions. The sensitivities are obtained by using the package PANDA [7]. By comparing the results with our previous investigations in [8] [9], it can be noted that the WS sensitivity behavior of the special second order sections is very similar to that of the corresponding first order sections but with the symmetry around the frequency f = 0.5. It is clearly seen that there exists a proper selection of the sections for every given TF pole position because of the significant difference between the maximal values of the sensitivities (in some cases it can reach more than 100 times especially for the poles near ±1).



(*a*) $b_{MH1\text{-}2}$ = 0.18654



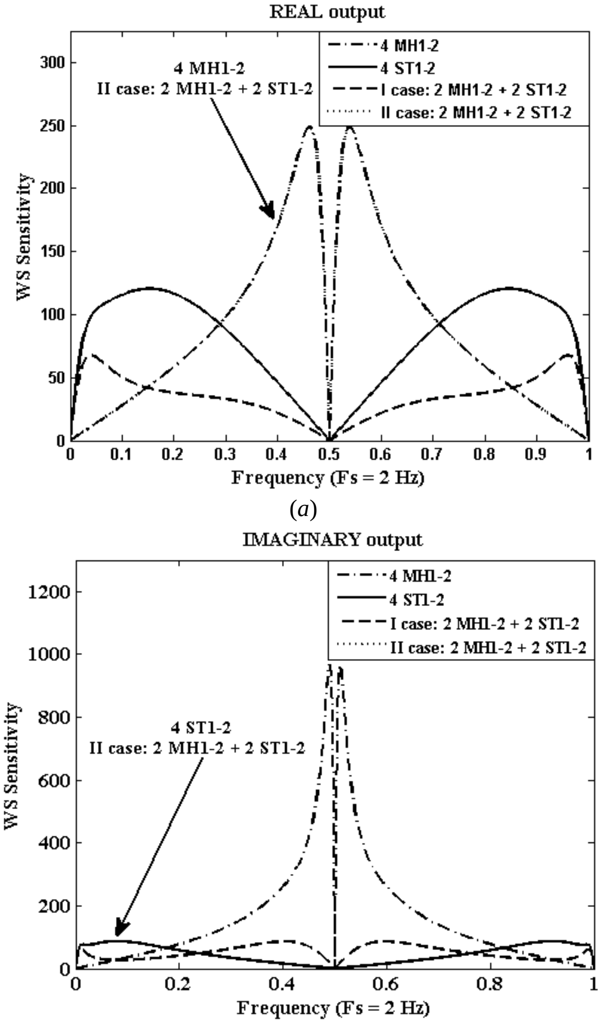
(*b*) $b_{MH1\text{-}2}$ = 0.94167

Fig. 4. Worst-case phase-sensitivities of second-order allpass sections (Fig. 3) for two different TF poles positions.

V. OVERALL SENSITIVITY INVESTIGATIONS

In order to estimate how the proper choice of the special sections will affect the behavior of the HT realization in a limited word-length environment, we have designed and investigated a ninth order HT having the TF poles positions given in Fig. 1 (the initial elliptic half-band filter specifications are: passband frequency F_p = 0.24 and stopband attenuation δ_s = 0.01 (R_s = 40 dB), producing $\Delta\varphi_{max}$ = 1.15º).

Then, we have designed 4 different HT realizations (Fig. 2). The first one was realized using the standard way (using only MH1-2 sections) and it is marked in the figures as "4MH1-2". The allpass sections selection for the other realizations is based on the sensitivity minimization of the individual sections depending on their poles positions. Thus, in the second HT realization (denoted with "4 ST1-2") four ST1-2 sections were used. In the third and fourth implementations, two MH1-2 and two ST1-2 sections have been selected. In the first case, we have a special section of each type in every branch of the realization, while in the second case – two MH1-2 sections are used in the upper branch (the real output) and two ST1-2 sections – for the imaginary output. The results for the overall sensitivity of the two branches are shown in Fig. 5.



(*a*)

(*b*)

Fig. 5. Worst-case phase-sensitivities of the HT (Fig. 2) realized with different sets of allpass sections (for a 9-th order HT).

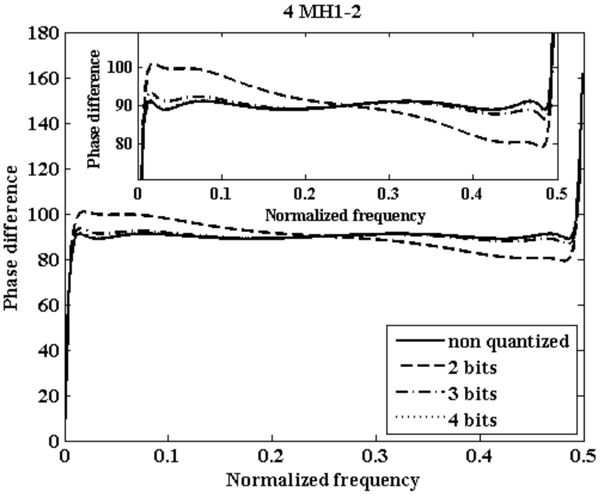
It appeared that the best configuration is with two MH1-2 and two ST1-2 sections, each in every branch (I case), providing the lowest overall sensitivity in both paths.

VI. INVESTIGATION OF THE INFLUENCE OF THE SECTIONS COMBINATIONS IN THE BRANCHES

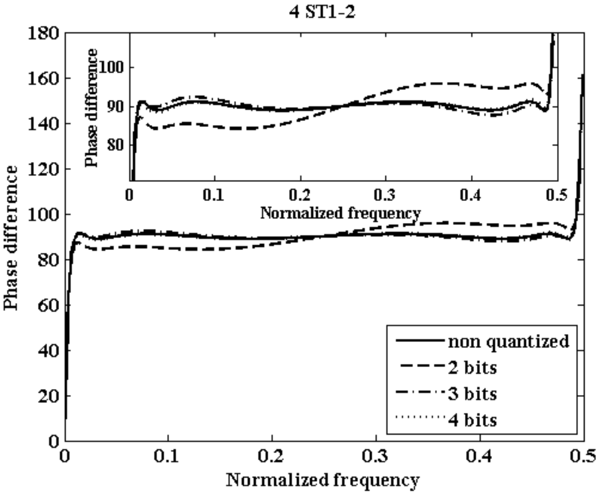
The phase difference between the two outputs in Fig. 2 will not be exactly 90º. Over some frequency range (narrower than half of the sampling frequency) it will alternate around this value with amplitudes $\Delta\varphi_{max}$ depending ideally only on the selected value of δ_s Eq. (2), but in reality – also on the design accuracy and on the parasitic effects of the digital realization. These additional deviations should be kept as lower as possible mainly by reducing the influence of the parasitic effects (by minimizing the sensitivities to the variations of the multiplier coefficients values). It will appear from what follows, that it might not be an easy straightforward procedure.

The accuracies of the HT realizations (the phase difference between the two branches) in a limited word-length environment are compared in Fig. 6. Based on the results shown in Fig. 5, it is natural to have a high sensitivity (to small changes in the two branches) of the phase difference between the two outputs in Fig. 2 for 4 MH1-2 HT realization, but the results shown in Fig. 6a are quite surprising, compared to these in Fig. 6b,c,d (with minimized sensitivity). We suppose that this might be an effect due to some internal compensation between the parasitic effects in the branches, explained with the different signs of the sensitivities. The worst-case sensitivity WS, used in our investigations, is not able to reveal these mutual compensations, because it is eliminating the signs of the individual sensitivities.

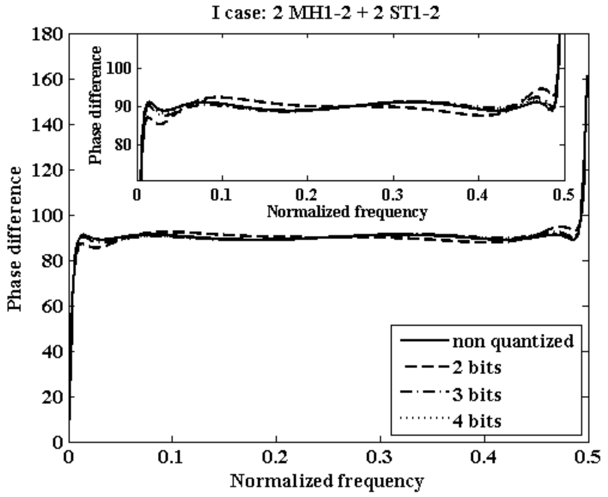
The highest accuracy, as it is shown in Fig. 6, is achieved when we have two MH1-2 and two ST1-2 sections each in every branch (I case) of the HT. In this case, the selection of the sections and their placement in the branches are made under the above mentioned observations.



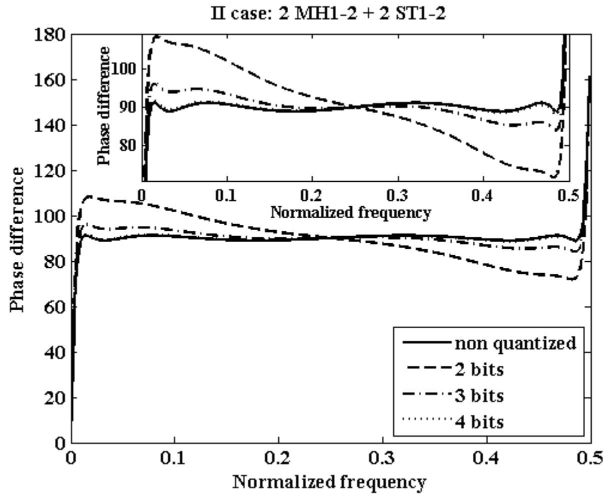
(*a*) using 4 *MH*1-2 sections



(*b*) using 4 *ST*1-2 sections



(*c*) using 2 *MH*1-2 and 2 *ST*1-2 sections - I case



(*d*) using 2 *MH*1-2 and 2 *ST*1-2 sections - II case

Fig. 6. Word-length dependence of the accuracy of the HT phase difference for realizations with different allpass sections.

As it can be seen after quantization to 2 bit (in CSD code) not only the fluctuations of the phase difference in Fig. 6a,b,d are growing very much above the ideal, but the range of frequencies over which this difference is approximately constant, is sharply reduced, while in Fig. 6c these parameters are practically unchanged.

The main conclusion of these investigations is that besides the sensitivity minimization, an additional step, consisting of a study of all possible combinations of the selected allpass sections within the branches, has to be introduced. A more general solution of this problem will be a derivation of a formula about the sensitivity of the phase quadrature to the changes of the multipliers' values, but it may appear to be a very difficult task.

VII. LOW-SENSITIVITY DESIGN PROCEDURE

Taking into account all results so obtained, we propose the following design procedure:

1. Obtain $H_{HT}(z)$ Eq. (5) by applying the standard design procedure from Sect. 2.
2. Decompose the TFs $A_1(z^{-2})$ and $A_2(z^{-2})$ to special second-order allpass TFs and find where their poles are situated.
3. Select (from Fig. 3) or develop new allpass sections realizing each couple of poles with the lowest sensitivity and verify this by sensitivity studies as these in Fig. 4.
4. Investigate the overall sensitivities in the two branches of Fig. 2 for all possible combinations of the selected allpass sections realizations in order to select the best set.
5. In case of a very high accuracy design, verify the selection by simulating the structure in a limited word-length environment (as in Fig. 6).

We have applied this procedure for different sets of specifications and it was always possible to find an implementation clearly outperforming all the others as the case in Fig. 6 c.

VIII. CONCLUSION

A new approach to improve the accuracy of the allpass based Hilbert transformers (realized as two parallel branches) through sensitivity minimizations of each individual special second-order allpass section in the cascade realizations of the two branches was proposed in this paper. The design procedure is simple and straightforward, without iterative and complicated optimization steps and is achieving accuracy of realizations close to the ideal case (nonquantized coefficients). The low sensitivities so attained permit also a very short coefficients word-length, a higher processing speed and lower power consumption.

REFERENCES

[1] C. S. Turner, "An efficient analytic signal generator", Signal Proc. Magazine, vol. 23, pp. 91-94, July 2009.

[2] C. B. Rorabaugh, "Notes on Digital signal processing: practical recipes for design, analysis, and implementation", Notes 58-66, Prentice Hall, 2011.

[3] P. A. Regalia, "Special Filter Designs", Chapter 13 in the book *Handbook for Digital Signal Processing* (Editors S. K. Mitra and J. F. Kaiser), John Wiley & Sons, NY, pp. 909-931, 1993.

[4] L. Milic, J. Certic, M. Lutovac, "A class of FRM-based allpass digital filters with applications in half-band filters and Hilbert transformers", Proc. Intern. Conf. on Green Circuits and Syst. (ICGCS)'2010, Shanghai, China, pp. 273-278, June 2010.

[5] S. Mitra, *Digital signal processing: A computer based approach*, McGraw-Hill, 2006.

[6] G. Stoyanov and H. Clausert, "A comparative study of first order digital all-pass structures", Frequenz, vol. 48, No 9/10, pp. 221-226, Sept./Oct. 1994.

[7] H. Sugino and A. Nishihara, "Frequency-domain simulator of digital networks from the structural description", Trans. of the IEICE of Japan, vol. E73, No.11, pp. 1804-1806, Nov. 1990.

[8] G. Stoyanov, Z. Nikolova, K. Ivanova, V. Anzova, "Design and realization of efficient IIR digital filter structures based on sensitivity minimizations", Proc. 8th IEEE Conf. TELSIKS'2007, Nish, Serbia, vol. 1, pp. 299–308, Sept. 2007.

[9] G. Stoyanov, K. Nikolova, and M. Kawamata, "Low-sensitivity design of allpass based fractional delay digital filters", Chapter 7 in the book *Digital filters*, F. P. Márquez (Ed.), Intech Publ. House, pp. 155–178, 2011.