Modulated bandpass Farrow Decimators and Interpolators

Djordje Babic[1] and Vesa Lehtinen [2]

Abstract – **The Farrow structure provides a flexible way for adjustable filtering, including adjustable fractional-delay filters and sample rate conversion (SRC) by arbitrary, non-integer, factors. Recent publications have pointed out its suitability to bandpass SRC. When the passband centre is located above the input sample rate, the interpolator complexity becomes approximately proportional to the centre frequency. We propose a new construct, the modulated Farrow structure, which allows arbitrarily high centre frequencies without increasing the polynomial degree of the prototype Farrow filter. The modulating function is constructed as a low-order polynomial in order to avoid costly generation of trigonometric functions.**

Keywords – **Digital Filters, Farrow Structure, Decimators, Interpolators, Band-pass Filtering.**

I. INTRODUCTION

Discrete-time filter structures are used in many applications to interpolate new sample values at arbitrary points between the discrete-time input samples [1], [2]. In those cases, it is beneficial to use polynomial-based interpolation filters. The advantage of the above system lies in the fact that the actual implementation can be efficiently performed by using the Farrow structure [1] or its modifications [2], [3]. In [2], [3], several design methods for polynomial-based interpolation filters have been developed in the time and in frequency domain. Using the frequency domain approach, it is possible to design the polynomial-based filter realized in the form of Farrow structure with an arbitrary baseband (zero center frequency) frequency response.

However, there are some applications where there is a need for a bandpass interpolation. Some example applications are: bandpass sampling, bandpass sampling rate conversion, rational filter banks, etc [4], [5]. It has been shown in [6], that if the center frequency of the bandpass interpolation filter is smaller than the half of the sample rate, for the given passband and stop-band requirements, the filter order does not increase compared to the baseband filter with the same requirements. The design methods presented in [2] and [3] can be effectively used for the design of bandpass polynomial interpolation filters with minor modifications [6].

However, when the center frequency is (far) higher than the input sample rate, the polynomial degree would become approximately proportional to the centre frequency. Therefore, we propose a new construct, the modulated Farrow structure, which allows arbitrarily high centre frequencies without increasing the polynomial degree of the lowpass prototype Farrow filter. The modulating function is constructed as a low-order polynomial in order to avoid costly generation of trigonometric functions.

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II. POLYNOMIAL MODULATION

Bandpass filters can be implemented with lowpass filters as building blocks by applying frequency translation to the signal before and after filtering which is equivalent to multiplying the impulse response by cosine whose frequency is equal to the center frequency of the bandpass filter [4]. While this permits arbitrary and variable centre frequencies, generation of costly cosine function is needed, this increases the implementation complexity. The same principle can be applied to the case of polynomial-based filter, as well.

In order to minimize the implementation complexity, it may be beneficial to approximate the modulating function with a piecewise polynomial. The modulating polynomial is chosen first, and the modulation is taken into account in the optimized subfilter coefficients. The non-idealities of the modulating function can be compensated by the filter. This eliminates the need for costly (co)sine generation.

Using Taylor polynomials a function can be approximated as closely as desired by a polynomial provided that the function posses sufficient number of derivatives [7], [8]. Based on the definition, the modulating cosine function $\cos(\omega_0 t)$ can be approximated with following polynomial:

$$\cos(\omega_0 t) = \sum_{j=0}^{j=J_a} (-1)^j \left[\cos(\omega_0 a)\frac{(\omega_0 t)^{2j}}{(2j)!} - \sin(\omega_0 a)\frac{(\omega_0 t)^{2(j+1)}}{(2j+1)!} \right]. \quad (1)$$

The above equation can be expressed in more suitable way for practical realization:

$$\cos(\omega_0 t) = \sum_{m=0}^{M_a} g_m t^m \ , \quad (2)$$

where

$$g_m = \begin{cases} (-1)^{m/2} \cos(\omega_0 a)\dfrac{(\omega_0)^m}{(m)!} & \text{for } m \text{ even, and} \\ -(-1)^{(m-1)/2} \sin(\omega_0 a)\dfrac{(\omega_0)^m}{(m)!} & \text{for } m \text{ odd,} \end{cases} \quad (3)$$

$M_a=2*J_a+1$ is a polynomial degree which is related to the approximation error, and $t\in[0,T)$. Here, T is the input sampling interval T_{in} in the interpolation case, and output sampling interval T_{out} in decimation case. Thus, we have obtained a polynomial that approximates the cosine function over one polynomial segment. It is possible to generalize the approximation of the cosine over a longer interval by making it a piecewise having N polynomial segments, and in each segment the order of the polynomial is M_a. The piecewise polynomial approximation of the cosine can be written as:

$$\cos(\omega_0 t) = \sum_{n=0}^{N} \sum_{m=0}^{M_a} b_m(n) t^m \,, \tag{4}$$

where

$$b_m = \begin{cases} (-1)^{m/2} \cos(\omega_0 a(n)) \dfrac{(\omega_0)^m}{(m)!} & \text{for } m \text{ even, and} \\ -(-1)^{(m-1)/2} \sin(\omega_0 a(n)) \dfrac{(\omega_0)^m}{(m)!} & \text{for } m \text{ odd.} \end{cases} \tag{5}$$

Here, $a(n)$ determines the initial phase of the approximated cosine function in each polynomial segment. The value of $a(n)$ can be selected in such a manner that overall approximated cosine function is symmetric around $N/2$, $a(n)=n+0.5$, for $n=0, 1 \ldots N-1$. In this way, the impulse response of modulated lowpass prototype filter is also symmetric. In the actual implementation, the symmetry can be exploited in a similar way as for polynomial-based filters, the polynomial in t can be transformed to polynomial in $(2t-1)$ thus we obtain

$$\cos(\omega_0 t) = \sum_{n=0}^{N} \sum_{m=0}^{M_a} b_m(n)(2t-1)^m = \sum_{n=0}^{N} \sum_{m=0}^{M_a} b_m(n)\, f_m(n,T,t) \,. \tag{6}$$

The modulated impulse response of the polynomial based filter is obtained by multiplying the impulse response of the lowpass prototype filter by cosine:

$$h_m(t) = h_a(t) \cos(\omega_0 t), \tag{7}$$

yielding to:

$$h_m(t) = \sum_{n=0}^{N} \left[\sum_{m=0}^{M} c_m(n) f_m(n,T,t) \sum_{m_1=0}^{M_a} b_{m_1}(n)\, f_{m_1}(n,T,t) \right]. \tag{8}$$

The modulated impulse response is a piecewise polynomial with the same number of polynomial segments as the lowpass prototype filter and polynomial order of $M+M_a$ in each segment. The polynomial order increases as the central frequency of the passband increases. Furthermore, the obtained bandpass filter is not configurable, as the new set of coefficients is calculated for different value of passband central frequency.

If we calculate the output sample according to hybrid analogue/digital model, the lth output sample can be expressed, after some manipulations, in the following form [2], [3]:

$$y(l) = \sum_{n=0}^{N-1} \left[\sum_{m=0}^{M} x(n_l - n + \frac{N}{2}) c_m(n)(2\mu_l - 1)^m \sum_{m_1=0}^{M_a} b_{m_1}(n)(2\mu_l - 1)^{m_1} \right] \tag{9}$$

where

$$n_l = \lfloor l T_{out} / T_{in} \rfloor \text{ and } \mu_l = l T_{out} / T_{in} - n_l \,. \tag{10}$$

Alternatively, the lth output sample can be expressed as:

$$y(l) = \sum_{m=0}^{M} \sum_{n=0}^{N} \eta_l(n) x(n_l - n + N/2) c_m(n)(2\mu_l - 1)^m \,, \tag{11}$$

with

$$\eta_l(n) = \sum_{m=0}^{M_a} b_m(n)(2\mu_l - 1)^m \,. \tag{12}$$

Here $\eta_l(n)$ is a modulating term. There are altogether N modulating terms $\eta_l(n)$, one for each polynomial segment. Equations (12) and (13) can be used to define a bandpass modulated modified Farrow structure, which is shown in Fig. 1. The structure can be used for different passband central frequencies, with a single prototype filter.

III. Special cases of modulated Farrow structure

There are several special cases of polynomial modulation, in which the process of building the bandpass Farrow structure can be further simplified. These special cases are derived from mutual relation between the desired central frequency $f_0=\omega_0/2\pi$ and the sample frequency of the bandpass filter $F_s=1/T$. If the central frequency is an integral multiple of the sample frequency, integral multiple of the half or quarter of the sample frequency then the modulating term $\eta_l(n)$ has special values as shown in a sequel.

The most straightforward case of modulated Farrow structures is that with the modulating frequency of kF_s. All that is needed is to multiply the output of the lowpass Farrow structure with a cosine at kF_s, i.e., when the modulation frequency is an integral multiple of F_s, the cosine generation is simplified by:

$$b_m = \begin{cases} (-1)^{m/2} (-1)^k \dfrac{(\omega_0)^m}{(m)!} & \text{for } m \text{ even, and} \\ 0 & \text{for } m \text{ odd.} \end{cases} \tag{13}$$

Therefore, b_m depends only on polynomial order m, and integral multiple k, reducing (12) to

$$\eta_l = \sum_{m=0}^{M_a} b_m (2\mu_l - 1)^m \,. \tag{14}$$

Fig. 2. depicts the modulated Farrow structure for modulation frequencies kF_s. Impulse response modulation at kF_s is attained simply by implementing modulation of Farrow structure by η_l.

When the modulation frequency is an odd multiple of $F_s/2$, the modulating function has the same shape in every polynomial segment, but its sign alternates between adjacent segments. The calculation of the modulating term reduces to

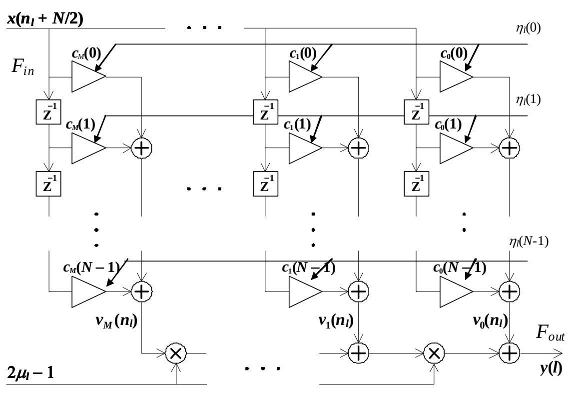

Fig. 1. The modulated modified Farrow structure

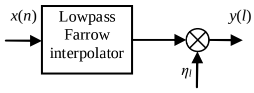

Fig. 2. The modulated Farrow structure for center frequencies kF_s.

$$b_m(n) = \begin{cases} 0 & \text{for } m \text{ even, and} \\ -(-1)^{(m-1)/2}(-1)^n(-1)^k \dfrac{(\omega_0)^m}{(m)!} & \text{for } m \text{ odd} \end{cases} \quad (15)$$

and, thus

$$\eta_l(n) = (-1)^n \sum_{m=0}^{M_a} b_m (2\mu_l - 1)^m \ . \quad (16)$$

Fig. 3. illustrates the modulated Farrow structure for modulation frequencies $(2k+1)\ F_s/2$. As above, modulation at $(2k+1)F_s/2$ is attained simply by multiplying by $\eta_l(n)$. and by alternating the sign of coefficients $c_m(n)$ for odd n.

Let us consider modulating the impulse response by a cosine at the frequency $(2k+1)F_s/4$. In each polynomial segment, the modulating function has a similar shape:(i) between the nth and $(n+2)$th segment, the shape is the same but the signs are opposite; (ii) adjacent segments are time-domain mirror images of each other with the same or opposite signs. The modulating functions can be written as

$$b_m(n) = \begin{cases} (-1)^{m/2}\cos((2k+1)(n+\frac{1}{2})\frac{\pi}{2})\dfrac{(\omega_0)^m}{(m)!}, & \text{for } m \text{ even,} \\ -(-1)^{(m-1)/2}\sin((2k+1)(n+\frac{1}{2})\frac{\pi}{2})\dfrac{(\omega_0)^m}{(m)!}, & \text{for } m \text{ odd.} \end{cases} \quad (17)$$

After applying series of trigonometric identities, the coefficients $b_m(n)$ are reduced to:

$$b_m(n) = \begin{cases} (-1)^{\frac{m}{2}}(-1)^{\frac{(2k+1)n-1}{2}}\sin((2k+1)\frac{\pi}{4})\dfrac{(\omega_0)^m}{(m)!}, & \text{for } m \text{ even,} \\ (-1)^{\frac{m+1}{2}}(-1)^{\frac{(2k+1)n-1}{2}}\cos((2k+1)\frac{\pi}{4})\dfrac{(\omega_0)^m}{(m)!}, & \text{for } m \text{ odd,} \end{cases} \quad (18)$$

for odd n, and

$$b_m(n) = \begin{cases} (-1)^{\frac{m}{2}}(-1)^{\frac{(2k+1)n}{2}}\cos((2k+1)\frac{\pi}{4})\dfrac{(\omega_0)^m}{(m)!} & \text{for } m \text{ even,} \\ (-1)^{\frac{m+1}{2}}(-1)^{\frac{(2k+1)n}{2}}\sin((2k+1)\frac{\pi}{4})\dfrac{(\omega_0)^m}{(m)!} & \text{for } m \text{ odd.} \end{cases} \quad (19)$$

for even n. These coefficients can be further simplified based on the value of modulating integer k as shown in a sequel: for n even, k even in (20); for n even, k odd in (21); for n odd, k even in (22); for n odd, k odd in (23);

$$b_m(n) = \begin{cases} (-1)^{m/2}(-1)^{\frac{(2k+1)n}{2}}(-1)^{\frac{k}{2}}\cos(\frac{\pi}{4})\dfrac{(\omega_0)^m}{(m)!} & \text{for } m \text{ even,} \\ -(-1)^{(m-1)/2}(-1)^{\frac{(2k+1)n}{2}}(-1)^{\frac{k}{2}}\sin(\frac{\pi}{4})\dfrac{(\omega_0)^m}{(m)!} & \text{for } m \text{ odd.} \end{cases} \quad (20)$$

$$b_m(n) = \begin{cases} (-1)^{m/2}(-1)^{\frac{(2k+1)n}{2}}(-1)^{\frac{k-1}{2}}\sin(\frac{\pi}{4})\dfrac{(\omega_0)^m}{(m)!} & \text{for } m \text{ even,} \\ -(-1)^{(m-1)/2}(-1)^{\frac{(2k+1)n}{2}}(-1)^{\frac{k-1}{2}}\cos(\frac{\pi}{4})\dfrac{(\omega_0)^m}{(m)!} & \text{for } m \text{ odd.} \end{cases} \quad (21)$$

$$b_m(n) = \begin{cases} (-1)^{m/2}(-1)^{\frac{(2k+1)n-1}{2}(-1)}(-1)^{\frac{k}{2}}\sin(\frac{\pi}{4})\dfrac{(\omega_0)^m}{(m)!} & \text{for } m \text{ even,} \\ -(-1)^{(m-1)/2}(-1)^{\frac{(2k+1)n-1}{2}}(-1)^{\frac{k}{2}}\cos(\frac{\pi}{4})\dfrac{(\omega_0)^m}{(m)!} & \text{for } m \text{ odd.} \end{cases} \quad (22)$$

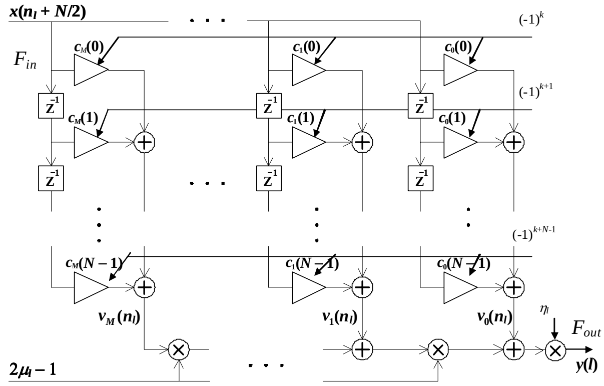

Fig. 3. The modulated Farrow structure for center frequencies $(2k+1)F_s/2$.

$$b_m(n) = \begin{cases} (-1)^{m/2}(-1)^{\frac{(2k+1)n-1}{2}}(-1)^{\frac{k-1}{2}}\cos(\frac{\pi}{4})\dfrac{(\omega_0)^m}{(m)!} & \text{for } m \text{ even} \\ -(-1)^{(m-1)/2}(-1)^{\frac{(2k+1)n-1}{2}}(-1)^{\frac{k-1}{2}}\sin(\frac{\pi}{4})\dfrac{(\omega_0)^m}{(m)!} & \text{for } m \text{ odd.} \end{cases} \quad (23)$$

It is possible to see from (20)-(23), that the values of coefficients $b_m(n)$ in this case depend only on the polynomial order m, and sign is determined by n, and k. The main consequence is that all $b_m(n)$s are the same for given m, while the sign alternates according to (20)-(23). In order to reduce the number of operations needed for modulation, it is beneficial to decompose the mth-degree subfilter into two polyphase branches:

$$\begin{aligned} c_{m0}(n) &= c_m(2n) \\ c_{m1}(n) &= c_m(2n+1). \end{aligned} \quad (24)$$

In this way, two interleaved Farrow structures are obtained:

$$C_m(z) = C_{m0}(z) + z^{-1}C_{m1}(z). \quad (25)$$

In each interleaved Farrow structure the sign of coefficients is determined by n and k, and the output is multiplied by corresponding value of η_{l0} or η_{l1}. Fig. 4. shows the modulated Farrow structure for modulation frequencies $(2k+1)F_s/4$.

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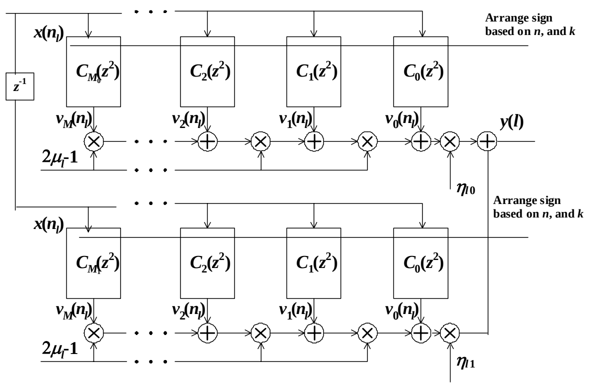

Fig. 4. The modulated Farrow structure for center frequencies $(2k+1)F_s/4$.

IV. Design Examples

In order to illustrate the effectiveness of the proposed modulation method for the design of bandpass Farrow structure we use several illustrative examples. The first step is the design of the lowpass prototype filter with passband edge $f_p=0.3F_s$, stopband edge $f_s=0.5F_s$, required stopband attenuation $A_s=60$dB, and passband tolerance $\delta_p=0.1$. The filter, with performance shown in Fig 5., is designed using minimax design of [2], having $N=14$ polynomial segments, and polynomial degree $M=4$.

The next step is to build the bandpass filter by modulating the obtained lowpass prototype. We use two example cases, with central frequencies of $3F_s/4$ and $2F_s$. The corresponding polynomial-based modulation function that approximates cosine has the same length $N=14$, and polynomial order $M_a=7$ for $3F_s/4$, and $M_a=19$ for $2F_s$. According to Section 3, there are 10 additional multiplications to produce modulation by frequency $3F_s/4$, and also 10 to produce modulation by frequency $2F_s$. Frequency domain performances of modulated bandpass filters are shown in Figs. 5 and 7 respectively. Though there is a slight degradation of performance in stopband, the filtering requirements can be met by over designing the lowpass prototype filter.

V. Conclusions

We have presented a modulated Farrow structure, which allows band-pass realization with arbitrarily high centre frequencies without increasing the polynomial degree of the lowpass prototype Farrow filter. The modulating function is constructed as a low-order polynomial in order to avoid costly generation of trigonometric functions. The non-idealities of the modulating function are then mitigated by taking them into account when optimizing the filter coefficients.

Acknowledgements

This work was supported by the Serbian Ministry of Science under technology development projects: TR32028 – "Advanced Techniques for Efficient Use of Spectrum in Wireless Systems" and TR32023 – "Performance Optimization of Energy-efficient Computer and Communications Systems."

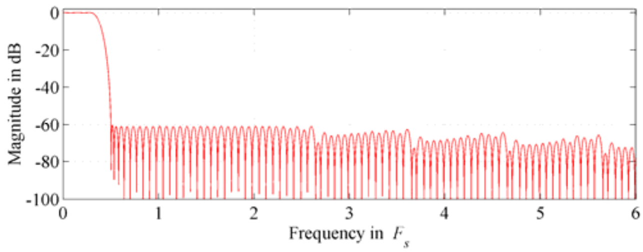

Fig. 5. The frequency response of the lowpass prototype.

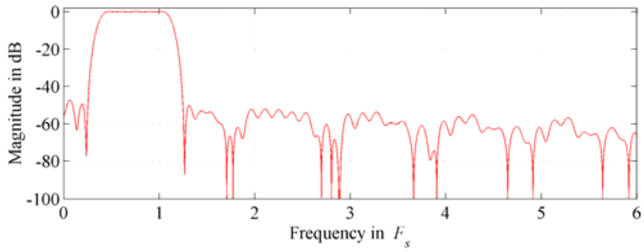

Fig. 6. The frequency response of the filter modulated by $3F_s/4$.

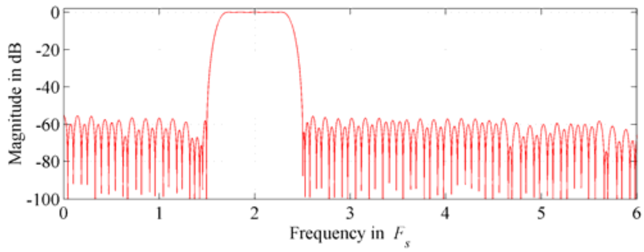

Fig. 7. The frequency response of the filter modulated by $2F_s$.

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