Computer Methods and New Values for Cut Set Catalan Numbers

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Abstract – **Improved methods to determine the values of the cut set Catalan numbers are presented. These methods require computing the number of special solutions of a system of linear inequalities. Moreover, the table of values for cut set Catalan numbers is completed up to n=29.**

Keywords – **cut set Catalan numbers, dissection of a convex polygon, triangulation.**

I. Introduction

Cut set Catalan numbers were introduced in [1] as a variation of the well-known classical Catalan numbers C_n. The Catalan number counts the number of dissections of a regular *(n+2)*-gon with labelled vertices by *n-1* non-intersecting diagonals into n triangles. The cut set Catalan numbers S_n are defined as follows: Let P_{n+2} be a regular convex *(n+2)*-gon with labelled vertices. Consider dissections of P_{n+2} by *n-1* non-intersecting diagonals into *n* triangles. Two such triangle sets $T'=\{T'_1, \ldots, T'_n\}$ and $T''=\{T''_1, \ldots, T''_n\}$ are said to be isomorphic if there exists a bijection $\varphi: T' \to T''$ such that T'_i and T''_i are congruent for $i=1, \ldots, n$. Then the cut set Catalan numbers S_n is defined to be the number of isomorphism classes of all such dissections of P_{n+2}.

In contrast to the classical Catalan numbers C_n, for S_n no explicit formula is known up to now. Some methods to determine S_n are presented in [1] and [2]. The new method in [4] is partly based on this results. We will now systemize the methods and analyze the ways to computing S_n. Moreover, the values of S_n are presented in [4] for $n \leq 29$.

The cut set Catalan numbers are registered in the On-Line Encyclopedia of Integer Sequences (OEIS, see [5]) with the sequence number A033961.

In [1] is proved that S_n equals the number of special solutions of a system of linear equalities. Then the method is improved in [2] and in [4]. Now we will present here these computer methods.

II. Special Linear System and S_N

A dissection Δ of a convex *(n+2)*-gon *P* means here a dissection by *n-1* non-intersecting diagonals into *n* triangles. The label $l_T(s)$ of a side *s* belonging to a triangle *T* in Δ is defined as the number of vertices of *P* between the endvertices *u* and v of *s* on that part of the boundary of *P* not containing the third vertex *w* of *T*. Note that the label sum of the three sides belonging to *T* equals *n-1*. If *a*, *b* and *c* are the labels of the sides of a triangle *T* in Δ, where $a \leq b \leq c$, then *T* is said to be of type *(a,b)*.

Let T_n be the set of the possible types of triangles:

$$T_n = \{(a,b) \in N_0 \times N_0 ; 0 \leq a \leq b \leq (n-1-a)/2\}. \qquad (1)$$

We use $\lambda_{a,b}$ for the number of triangles of type *(a,b)* in Δ. It is easy to see that the upper bound $\lambda_{a,b} \leq \lfloor (n+2)/(a+b+2) \rfloor$ holds for all $(a,b) \in T_n$.

A triangle of type *(a,b)* is called a boundary triangle if *a=0*, and an inner triangle if $a \geq 1$. The set of types for the inner triangles is

$$I_n = \{(a,b) \in N_0 \times N_0 \mid 1 \leq a \leq b \leq (n-1-a)/2\}. \qquad (2)$$

A central triangle in Δ is a triangle of type *(a,b)* where $a+b \geq n/2-1$. A central triangle can be only of certain types and the number of the central triangles in one dissection can be only one or two.

Since every diagonal *d* in Δ is a side of two triangles, the sum of the two labels of *d* equals *n*. Thus, the number of occurrences of the label *i* in Δ is equal to the number of occurrences of the label *n-i* in Δ for *i=1, …, n-1*. These equalities and the requirement of the central triangle imply the following system of linear inequalities.

For $(a,b) \in I_n$, $i = 0, \ldots \lfloor (n-3)/2 \rfloor$ let

$$\zeta_{a,b}(i) = \begin{cases} 0, & if\ a \leq i\ and\ b > i\ or\ if\ a+b \leq i-1, \\ 1, & if\ a \geq i+1, \\ -1, & if\ a \leq i, b \leq i, and\ i \leq a+b < n-1-i, \\ -2, & if\ a \leq i, b \leq i, and\ a+b \geq n-1-i. \end{cases} \qquad (3)$$

Then the cut set Catalan number S_n equals the number of solutions in nonnegative integers of the following system of $\lfloor (n+1)/2 \rfloor$ linear inequalities with the $\lfloor ((n-1)^2+3)/12 \rfloor$ variables $\lambda_{a,b} \leq \lfloor (n+2)/(a+b+2) \rfloor$, $(a,b) \in I_n$:

$$\begin{aligned} 0 &\leq 2 + \sum_{(a,b)\in I_n} \zeta_{a,b}(i)\lambda_{a,b} \quad for\ i = 0, \ldots, \lfloor (n-3)/2 \rfloor, \\ 0 &\leq 2 - \sum_{\substack{(a,b)\in I_n \\ a+b=n/2-1}} \lambda_{a,b} - 2\sum_{\substack{(a,b)\in I_n \\ a+b\geq n/2}} \lambda_{a,b} \quad if\ n\ is\ even, \\ 0 &\leq 1 - \sum_{\substack{(a,b)\in I_n \\ a+b\geq (n-1)/2}} \lambda_{a,b} \quad if\ n\ is\ odd. \end{aligned} \qquad (4)$$

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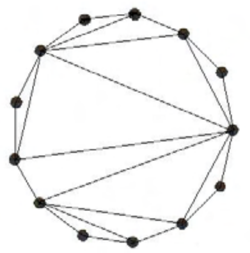

Fig. 1. A dissection of 13-gon

Thus, the values of S_n have been determined for $n\leq 29$ without the need for examination of dissections of a polygon. The values for S_n known up to now are given in Table I.

TABLE I
THE VALUES OF S_N

n	2	3	4	5	6	7	8	9	10	11	12	13
S_n	1	1	2	2	4	6	11	17	35	57	115	203

N	14	15	16	17	18	19	20
S_n	412	745	1546	2838	5901	11154	23255

N	21	22	23	24	25	26
S_n	44263	93169	179214	377441	733151	1547068

n	27	28	29
S_n	3020878	6399874	12543862

In the system of linear inequalities two inequalities, for $i=0$ and $i=1$, can be deleted because they follow from the other inequalities. For example the system of linear inequalities for $n=11$ gives the value $S_{11}=57$:

$$\begin{array}{lll} 0 & \leq & 2-\lambda_{1,1}-\lambda_{1,2}-\lambda_{2,2}+\lambda_{3,3} \\ 0 & \leq & 2-\lambda_{1,2}-\lambda_{1,3}-\lambda_{2,2}-\lambda_{2,3}-\lambda_{3,3} \\ 0 & \leq & 2-\lambda_{1,3}-\lambda_{1,4}-\lambda_{2,2}-\lambda_{2,3}-2\lambda_{2,4}-2\lambda_{3,3} \\ 0 & \leq & 1-\lambda_{1,4}-\lambda_{2,3}-\lambda_{2,4}-\lambda_{3,3} \end{array} . \quad (5)$$

III. THE METHODS FOR COMPUTING S_N

The determination of values of S_n is doing with the aid of computer programs in C. The first program is already presented in [1] and it gives S_n for $n\leq 19$. It counts the number of solutions of a linear system of equalities. In [2] the method is improved and we succeeded in evaluating S_n for $20\leq n\leq 25$. This program counts the number of solutions of a linear system of inequalities from Eq. (4) in nonnegative integers. A equivalent set of the system is constructed and it has a smaller number of inequalities. At this time the evaluation for S_{25} took about 100 hours. By the new values S_n for $26\leq n\leq 29$ the algorithm is once again improved [4]. The count the number of solutions we proceeded here in two steps.

Step 1. We first considered the subsystem of a system in Eq. (4) consisting of the inequalities concerning $i=\lfloor (n-1)/3\rfloor,\ldots,\lfloor (n-3)/2\rfloor$ and the last inequality. From $i\geq \lfloor (n-1)/3\rfloor$ follow $a\leq \lfloor (n-1)/3\rfloor$, $a<i$ and $\zeta_{a,b}(i)\leq 0$. We used a computer algebra system to determine all solutions of the subsystem in nonnegative integers. This step could be done in a few seconds.

Step 2. Using a simple algorithm, for each solution of the subsystem we counted the number of possibilities to choose the remaining variables $\lambda_{a,b}$ such that the inequalities for $i=0,\ldots,\lfloor (n-1)/3\rfloor -1$ and $0\leq \lambda_{a,b}\leq \lfloor (n+2)/(a+b+2)\rfloor$ are satisfied. For n=28 and n=29 the calculation were parallelized and distributed among more computers.

The program determines not only S_n but also all the relevant dissections of a polygon into triangles. The solutions of the linear system Eq. (4) give the numbers of the inner triangles and for every dissection the numbers of boundary triangles are determined from a system of equalities, see [2]. Haw can be composed the dissection corresponding to this set of triangles? We start with the central triangle or two central triangles (possible for n even). For every diagonal we draw another triangle (every diagonal is side in two triangles and has two labels). This procedure ends successfully because the relations between the labels in the system of inequalities.

This is one of the solutions of a system from Eq. (5): $\lambda_{0,0}=5$, $\lambda_{1,2}=\lambda_{0,1}=2$, $\lambda_{1,4}=\lambda_{0,4}=1$,$\lambda_{1,1}=\lambda_{1,3}=\lambda_{2,2}=\lambda_{2,3}=\lambda_{2,4}=\lambda_{3,3}=\lambda_{0,2}=\lambda_{0,3}=0$. The corresponding dissection is given in Fig. 1.

IV. CONCLUSION

At the time we work on the systems of inequalities for S_{30} and S_{31}. Based on a special classification of the dissections and the recursion formula from [2] we hope to get new values.

The cut set Catalan numbers are related to the other research areas: the maximum and minimum number of incongruent triangles in a dissection of regular convex polygon, see [3], the number of dissections by given number of triangle types, the numbers $S_n(i)$ of cut set Catalan numbers with inner triangles etc.

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