The level crossing rate of the ratio of product of two k-μ random variables and k-μ random variable

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Abstract –**In this paper the ratio of product of two k-μ random variables and k-μ random variable, but also, the ratio of k-μ random variable and product of two k-μ random variables are considered. The joint probability density function of the ratios and its first derivatives are calculated. Furthermore, the average level crossing rate of those ratios are evaluated. Numerical results are graphically presented to show the influence of parameters of fading on system performance.**

Keywords –**Level crossing rate (LCR), k-μ distribution.**

I. INTRODUCTION

The ratio of random variables is important performance in analysis of wireless communication systems operating over fading channels. Statistics of the ratio of random variables enables evaluation of outage probability, bit error probability and capacity of wireless communication systems. Signal envelope and co channel envelope experience small and large scale variation in the presence of fading. Short term fading is result of multipath propagation due to refraction, reflection, diffraction and scattering of radio wave. Long term fading is result of large obstacles and large deviation in terrain profile between transmitter and receiver. The co channel interference is interfering signal at the same frequency. Signals from two or more channels of different location and at the same frequency interfere. In interference limited environment level of co channel interference is sufficiently high as compared with thermal noise, so the noise can be ignored in performance analysis. In this channel the ratio of signal and co channel interference envelope is important performance measure of communication systems. In this paper, the ratio of product of two random variables and random variable is considered. In paper [1], the ratio of product of two α-μ random variables and α-μ random variable is calculated and probability density function is obtained. By using this result, bit error probability and outage probability of wireless communication system over composite α-μ multipath fading in the presence of co channel interference subjected to α-μ multipath fading can be evaluated. The product of two random variables in the nominator of the ratio can represent desired signal envelope affected simultaneously to two multipath fadings. The random variable in denominator of the ratio can represent co channel interference envelope subjected to multipath fading.

In paper [2], the ratio of random variable and product of two random variables is analyzed. The probability density function and cumulative distribution function of ratio of α-μ random variable and the product of two α-μ random variables are evaluated. Random variable in nominator of the ratio can represent desired signal in the presence of multipath fading. The product of two random variables in the denominator of the ratio can represent co channel interference envelope which suffer simultaneously from two multipath fadings. The expression for probability density function can be used for evaluation of bit error probability, outage probability, and capacity of wireless communication system operating over multipath fading channel in the presence of co channel interference subjected simultaneously to two multipath fadings.

In paper [3], the ratio of two product of two random variables is considered. The probability density function and cumulative distribution function of the ratio of two products of two α-μ random variables are calculated. The product of two random variables in the nominator of the ratio can represent desired signal envelope subjected simultaneously to two multipath fadings. The product of two random variables in denominator of the ratio can represent co channel envelope affected simultaneously to two multipath fadings. The expression for probability density function can be used for evaluation of bit error probability, outage probability, and capacity of wireless communication systems.

In this paper, the ratio of product of two k-μ random variables and k-μ random variable, and the ratio of k-μ random variables and product of two k-μ random variables is considered. The average level crossing rate of the ratios of random variables is calculated. The k-μ random variable has two parameters. The parameter k is related with dominant component envelope of fading. The parameter μ is associated with the number of clusters of scattering wave. The k-μ distribution describes small scale signal envelope fluctuation in multipath fading, linear and line-of-sight environments. The k-μ distribution is general distribution and Rayleigh, Rice and Nakagami-m distribution can be derived from k-μ distribution as special cases. By setting k=0, k-μ distribution reduces to Nakagami-m distribution. For μ=1, from k-μ distribution can be derived Rice distribution and for k=0 and μ=1, k-μ distribution reduces to Rayleigh distribution.

The results obtained in this paper can be used in performance analyzes of wireless communication system. The proposed communication system operates over composite $k-\mu$ multipath fading channel in interference limited environment. Co channel interference is affected to $k-\mu$ multipath fading. In interference limited environment the level of interference is

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sufficiently higher as compared to noise power so the interference of noise on system performance can be ignored. This paper is organized in the following sequence. In section II the ratio of the product of two k-μ random variables and k-μ random variable is calculated. In section III the level crossing rate of ratio of k-μ random variable and product of two k-μ random variables is evaluated. Section VI concludes the work.

II. Ratio of the Product of Two $k-\mu$ Random Variables and $k-\mu$ Random Variable

Random variables x, y and z follow k-μ distribution [4].

$$x = {x_1}^2 + {x_2}^2 + \cdots + {x_{2\mu}}^2 \tag{1}$$

$$y = {y_1}^2 + {y_2}^2 + \cdots + {y_{2\mu}}^2 \tag{2}$$

$$z = {z_1}^2 + {z_2}^2 + \cdots + {z_{2\mu}}^2 \tag{3}$$

Random variables $x_1, x_2, \ldots x_{2\mu}$, $y_1, y_2, \ldots y_{2\mu}$, and $z_1, z_2, \ldots z_{2\mu}$ have Gaussian distribution

$$P_{x_i}(x_i) = \frac{1}{\sqrt{2\pi}\,\delta_1} e^{-\frac{(x_i - A_1)^2}{2\,{\delta_1}^2}}, i = 1,2, \ldots 2\mu \tag{4}$$

$$P_{y_j}(y_j) = \frac{1}{\sqrt{2\pi}\,\delta_2} e^{-\frac{(y_j - A_2)^2}{2\,{\delta_2}^2}}, j = 1,2, \ldots 2\mu \tag{5}$$

$$P_{z_k}(z_k) = \frac{1}{\sqrt{2\pi}\,\delta_3} e^{-\frac{(z_k - A_3)^2}{2\,{\delta_3}^2}}, k = 1,2, \ldots 2\mu \tag{6}$$

The probability density functions of k-μ random variables x, y and z are

$$p_x(x) = \frac{2\mu(k+1)^{\frac{\mu+1}{2}}}{k^{\frac{\mu-1}{2}} e^{\mu k}\Omega_1} \left(\frac{x}{\Omega_1}\right)^{\mu} e^{-\mu(1+k)\left(\frac{x}{\Omega_1}\right)^2}$$

$$* \; I_{\mu-1}\left(2\mu\sqrt{k(1+k)}\,\frac{x}{\Omega_1}\right) \tag{7}$$

$$p_y(y) = \frac{2\mu(k+1)^{\frac{\mu+1}{2}}}{k^{\frac{\mu-1}{2}} e^{\mu k}\Omega_2} \left(\frac{y}{\Omega_2}\right)^{\mu} e^{-\mu(1+k)\left(\frac{y}{\Omega_2}\right)^2}$$

$$* \; I_{\mu-1}\left(2\mu\sqrt{k(1+k)}\,\frac{y}{\Omega_2}\right) \tag{8}$$

$$p_z(z) = \frac{2\mu(k+1)^{\frac{\mu+1}{2}}}{k^{\frac{\mu-1}{2}} e^{\mu k}\Omega_3} \left(\frac{z}{\Omega_3}\right)^{\mu} e^{-\mu(1+k)\left(\frac{z}{\Omega_3}\right)^2}$$

$$* \; I_{\mu-1}\left(2\mu\sqrt{k(1+k)}\,\frac{z}{\Omega_3}\right) \tag{9}$$

The ratio of the product of two k-μ random variables x and y and k-μ random variable z is

$$w = \frac{xy}{z}, x = \frac{wz}{y} \tag{7}$$

The first derivative of w is

$$\dot{w} = \frac{\dot{x}y}{z} + \frac{x\dot{y}}{z} - \frac{xy\dot{z}}{z^2} \tag{8}$$

The first derivative of random variables x, y and z are

$$\dot{x} = 2x_1\dot{x_1}, +2x_2\dot{x_2} + \cdots + 2x_{2\mu}\dot{x_{2\mu}} \tag{9}$$

$$\dot{y} = 2y_1\dot{y_1}, +2y_2\dot{y_2} + \cdots + 2x_{2\mu}\dot{y_{2\mu}} \tag{10}$$

$$\dot{z} = 2x_1\dot{z_1}, +2x_2\dot{z_2} + \cdots + 2x_{2\mu}\dot{z_{2\mu}} \tag{11}$$

The linear transformation of Gaussian random variables is Gaussian random variable. Therefore, random variables $\dot{x}, \dot{y}$ and $\dot{z}$ have conditional Gaussian probability density functions. The average values and variances of $\dot{x_1}, \dot{x_2}$ and $\dot{x_{2\mu}}$ are

$$\bar{\dot{x}}_1 = \bar{\dot{x}}_2 = \cdots = \bar{\dot{x}}_{2\mu} = 0 \tag{12}$$

$${\delta_{\dot{x_1}}}^2 = {\delta_{\dot{x_2}}}^2 = \cdots = {\delta_{\dot{x_{2\mu}}}}^2 = \pi^2\, 2{\delta_1}^2 {f_m}^2 = f_1^2 \tag{13}$$

The average values and variances of $\dot{y_1}, \dot{y}$ and $\dot{y_{2\mu}}$ are

$$\bar{\dot{y}}_1 = \bar{\dot{y}}_2 = \cdots = \bar{\dot{y}}_{2\mu} = 0 \tag{14}$$

$${\delta_{\dot{y_1}}}^2 = {\delta_{\dot{y_2}}}^2 = \cdots = {\delta_{\dot{y_{2\mu}}}}^2 = \pi^2\, 2{\delta_2}^2 {f_m}^2 = f_2^2 \tag{15}$$

The average values and variances of $\dot{z_1}, \dot{z_2}$ and $\dot{z_{2\mu}}$ are

$$\bar{\dot{z}}_1 = \bar{\dot{z}}_2 = \cdots = \bar{\dot{z}}_{2\mu} = 0 \tag{16}$$

$${\delta_{\dot{z_1}}}^2 = {\delta_{\dot{z_2}}}^2 = \cdots = {\delta_{\dot{z_{2\mu}}}}^2 = \pi^2\, 2{\delta_3}^2 {f_m}^2 = f_3^2 \tag{17}$$

The random variable $\dot{w}$ is

$$\dot{w} = \frac{2y}{z}\left(x_1\dot{x_1} + x_2\dot{x_2} + \cdots + x_{2\mu}\dot{x_{2\mu}}\right)$$

$$+ \frac{2x}{z}\left(y_1\dot{y_1} + y_2\dot{y_2} + \cdots + y_{2\mu}\dot{y_{2\mu}}\right)$$

$$- \frac{2xy}{z^2}\left(z_1\dot{z_1} + z_2\dot{z_2} + \cdots + z_{2\mu}\dot{z_{2\mu}}\right) \tag{18}$$

Random variable $\dot{w}$ follows conditional Gaussian distribution [5]. The mean value of $\dot{w}$ is zero. The variance of $\dot{w}$ is

$${\delta_{\dot{w}}}^2 = \frac{4y^2}{z^2}\left({x_1}^2 {\delta_{\dot{x_1}}}^2 + {x_2}^2 {\delta_{\dot{x_2}}}^2 + \cdots + {x_{2\mu}}^2 {\delta_{\dot{x_{2\mu}}}}^2\right)$$

$$+ \frac{4x^2}{z^2}\left({y_1}^2 {\delta_{\dot{y_1}}}^2 + {y_2}^2 {\delta_{\dot{y_2}}}^2 + \cdots + {y_{2\mu}}^2 {\delta_{\dot{y_{2\mu}}}}^2\right)$$

$$+\frac{4x^2y^2}{z^2}\left(z_1{}^2\,\delta_{\dot{z}_1}{}^2+z_2{}^2\,\delta_{\dot{z}_2}{}^2+\cdots+z_{2\mu}{}^2\,\delta_{\dot{z}_{2\mu}}{}^2\right)$$

$$\delta_{\dot{w}}{}^2=\frac{4w}{yz}\left(y^2f_1^2+wzf_2^2+wyf_3^2\right) \quad (19)$$

The conditional probability density function of $\dot{w}$ is

$$p_{\dot{w}}(\dot{w}/wyz)=\frac{1}{\sqrt{2\pi}}\frac{\sqrt{yz}}{2\sqrt{w}\sqrt{\left(y^2f_1^2+wzf_2^2+wyf_3^2\right)}}$$

$$*\ e^{-\frac{\dot{w}yz}{8w\left(y^2f_1^2+wzf_2^2+wyf_3^2\right)}} \quad (20)$$

The joint probability density function of $\dot{w}$, w, y and z is

$$p_{w\dot{w}yz}(w\dot{w}yz)=p_{\dot{w}}(\dot{w}/wyz)p_w(w/yz)p_y(y)p_z(z) \quad (21)$$

The conditional probability density function of w is

$$p_w(w/yz)=\left|\frac{dx}{dw}\right|p_x\left(\frac{wz}{y}\right)=\frac{z}{y}p_x\left(\frac{wz}{y}\right) \quad (22)$$

After substitution of (14), (15) and (16) in (13), the joint PDF of $w\dot{w}yz$ becomes

$$p_{w\dot{w}yz}(w\dot{w}yz)=p_{\dot{w}}(\dot{w}/wyz)\frac{z}{y}p_x\left(\frac{wz}{y}\right)p_y(y)p_z(z) \quad (23)$$

The joint probability density function of w and $\dot{w}$ is

$$p_{w\dot{w}}(w\dot{w})=\int_0^\infty dy\int_0^\infty dz\,p_{\dot{w}}(\dot{w}/wyz)$$

$$*\frac{z}{y}p_x\left(\frac{wz}{y}\right)p_y(y)p_z(z) \quad (24)$$

The level crossing rate of the ratio of product of two k-μ random variables and k-μ random variable is [4]

$$N_z=\int_0^\infty \dot{w}p_{w\dot{w}}(w\dot{w})\,d\dot{w} \quad (25)$$

$$=\int_0^\infty dy\int_0^\infty dz\frac{z}{y}\frac{1}{\sqrt{2\pi}}\frac{2\sqrt{w}}{\sqrt{yz}}\sqrt{\left(y^2f_1^2+wzf_2^2+wyf_3^2\right)}$$

$$*\,p_x\left(\frac{wz}{y}\right)p_y(y)p_z(z) \quad (26)$$

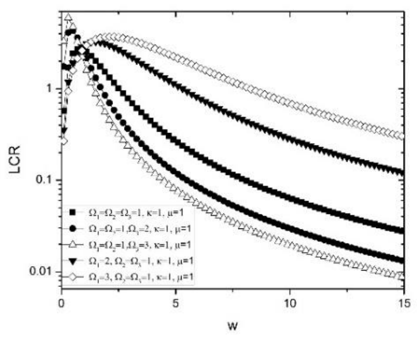

Fig.1 LCR for different *Ω₁, Ω₂*

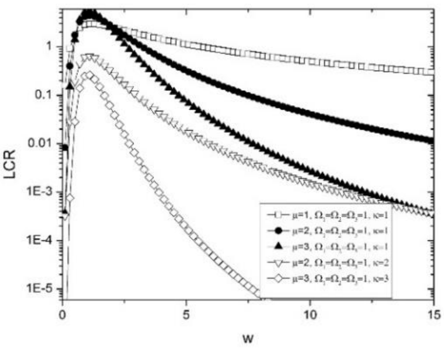

Fig.2 LCR for various parameters k and μ

In Figures 1 and 2, the average level crossing rate of the ratio of product of two k-μ random variables and k-μ random variable is presented in term of signal envelope. In figure 1, the influence of input signal power on average level crossing rate is shown. In Figure 2, the influence of parameter μ on average level crossing rate is shown. As parameter μ increases, for various signal envelope values, the average level crossing rate increases. Furthermore, as the parameter k increases, average level crossing rate, also increases.

III. Ratio of $k-\mu$ Random Variable and Product of Two $k-\mu$ Random Variables

In this section, the ratio of k-μ random variable and product of two k-μrandom variables is considered. The k-μ random variable in the nominator can represent desired signal envelope subjected to multipath fading. The product of k-μ random variables can represent cochannel interference envelope affected, simultaneously, to two k-μ fadings. The ratio is

$$w=\frac{x}{yz},x=wyz \quad (28)$$

The first derivative of z is now

$$\dot{w}=\frac{\dot{x}}{yz}-\frac{x\dot{y}}{y^2z}-\frac{x\dot{z}}{\dot{y}z^2}$$

$$=\frac{2y}{z}\left(x_1\dot{x}_1+x_2\dot{x}_2+\cdots+x_{2\mu}\dot{x}_{2\mu}\right)$$

$$-\frac{2x}{y^2z}\left(y_1\dot{y}_1+y_2\dot{y}_2+\cdots+y_{2\mu}\dot{y}_{2\mu}\right)$$

$$-\frac{2xy}{yz^2}\left(z_1\dot{z}_1+z_2\dot{z}_2+\cdots+z_{2\mu}\dot{z}_{2\mu}\right) \quad (29)$$

The mean of $\dot{w}$ is zero. The variance of $\dot{w}$ is now

$$\delta_{\dot{w}}{}^2=\frac{4w}{y^2z^2}\left(yzf_1^2+wyz^2f_2^2+wy^2zf_3^2\right) \quad (30)$$

The conditional Gaussian distribution of $\dot{w}$ is now

$$p_{\dot{w}}(\dot{w}/wyz) = \frac{1}{\sqrt{2\pi}} \frac{yz}{2\sqrt{w}\sqrt{\left(yzf_1^2+wyz^2f_2^2+wy^2zf_3^2\right)}} * e^{-\frac{\dot{w}yz}{8w\left(y^2f_1^2+wzf_2^2+wyf_3^2\right)}} \quad (31)$$

Similar mathematical apparatus is used like in (22)-(25) considering the ratio of k-μ random variable and product of two k-μ random variables.

The level crossing rate of the ratio of k-μ random variable and product of two k-μ random variables is

$$N_z = \int_0^\infty dy \int_0^\infty dz \frac{1}{\sqrt{2\pi}} 2\sqrt{w}\sqrt{\left(yzf_1^2 + wyz^2f_2^2 + wy^2zf_3^2\right)} * p_x\left(\frac{wz}{y}\right) p_y(y) p_z(z) \quad (32)$$

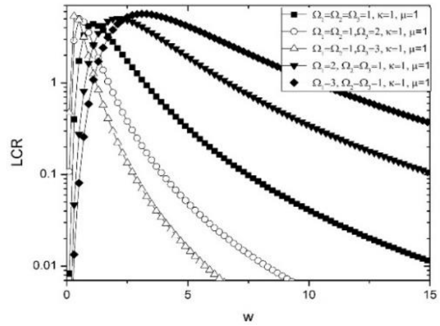

Fig.3 LCR for different Ω_1, Ω_2

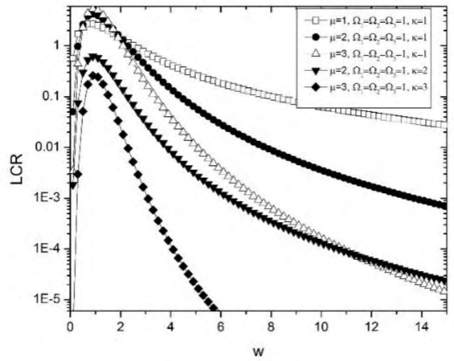

Fig.4 LCR for different parameters Ω_1, Ω_2, k and μ

In Figures 3 and 4, the average level crossing rate of the ratio of $k-\mu$ random variable and product of two $k-\mu$ random variables is presented versus output signal envelope of wireless communication system operating over $k-\mu$ multipath fading environment. In figure 3, the influence of input signal envelope power is shown. In figure 4 the influence of parameters k and μ on level crossing rate is presented.

IV. Conclusion

In this paper the ratio of product of two k-μ random variables and k-μ random variable, but also, the ratio of k-μ random variable and product of two k-μ random variables are considered. The joint probability density functions of the analyzed ratios and its first derivatives are derived. Also, level crossing rate of those ratios are calculated. Considering the ratio of product of two k-μ random variables and k-μ random variable, the product of two random variables in the nominator of the ratio can represent desired signal envelope affected simultaneously to two independent multipath k-μ fadings. Considering the ratio of k-μ random variable and k-μ random variables, the random variable in the nominator of the ratio can represent desired signal envelope affected simultaneously to two independent multipath k-μ fadings. The product of k-μ random variables in denominator of the ratio can represent cochannel interference envelope subjected to two multipath k-μ fadings. In interference limited, k-μ multipath fading line-of-sight environment, the ratio of product of two k-μ random variables and k-μ random variable, and also the ratio of k-μ random variables and product of two k-μ random variables can represent signal-to-interference envelopes ratios and can be used for calculation of system performances such as the outage probability, bit error probability and system capacity. In this paper the second order system performances are evaluated such as average level crossing rate, the joint probability density function and its first derivatives. This results are shown graphically for different parameters.

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