

# Inharmonicity of Two-Tones In Contra Octave of Upright Piano

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**Abstract** – The paper describes inharmonicity of musical instruments with strings. In the first part an algorithm for determination of coefficient of inharmonicity is shown, as well as algorithm for inharmonicity estimation and estimation of mean aliquote error of two-tone. In the second part paper shows results of algorithm's application in signal processing of tones in contra octave of upright piano "August Förster" manufactured in the year 1970. The results are given in graphics and in tables. An estimation of inharmonicity and medium aliquote error is given by comparative analysis made with Steinway pianos and Nordiska 1 and Straud upright pianos.

**Keywords** – Fundamental frequency, harmonic, inharmonic, piano.

## I. INTRODUCTION

Music theory defines basic characteristics of sound: a) lasting, b) intensity and c) color. The term color applies to sound in transcendental meaning which shows complexity of this sound feature. Sound source produces a sound with basic frequency (basic tone) and following tones (aliquotes of basic tone). Different number of present aliquotes (**lat. aliquoties**, few times) and their different relative volume within the frame of whole sound determine a sound color [1]. Aliquotes are also called partial tones or partials. Related to the frequency of basic tone  $f_0$  (fundamental frequency) aliquotes can have frequencies which are: a) integers (harmonics) and b) fractional multiplication of fundamental frequency (inharmonics) [2].

Instruments with strings produce tones by means of string vibration. If we talk about ideal string fixed at both ends, aliquotes are harmonious. Ideal string is to be understood as string with infinite large elasticity. However, in reality, there is not ideal string, but finite elasticity i.e. stiffness. Piano strings are tensed by strong force so their elasticity is reduced. As a consequence, frequency positions of aliquotes are at positions of non-integer multiplications of fundamental frequency. Logically, an instrument with such strings is not harmonious. Besides string's stiffness, inharmonicity of instrument is influenced by character of acoustic impedance of piano's resonator's board i.e. guitar's resonator's body [3]. Discrepancy from harmonicity, as a consequence gives less aliquote instruments. The tone produced on inharmonious instrument is not necessarily unpleasant. There is a statement in [3] that slightly inharmonious tone sounds in a way warmly.

Piano inharmonicity phenomena are described in many pa-

pers. Probably the oldest could be [4] and [5]. Paper [6] suggests formula which defines relation among frequency of aliquote  $f_k$  and inharmonicity coefficient  $\beta$  of vibrating string. Once we determine inharmonicity coefficient of strings that generate tone, we have determined inharmonicity coefficient of whole instrument. In [7] we see that piano and upright piano inharmonicity of bass range is in scope of  $50 \times 10^{-6}$  till  $600 \times 10^{-6}$ . In [8] measurements are performed and inharmonicity factor has been determined for bass-range strings: sub-contra octave (A0), contra-octave (E1 and A1) and great octave (E2 and A2) on Steinway D (grand piano), Steinway C (baby grand piano), Nordiska 1 (upright piano) and Straud (upright piano). This system of tones designation, known as Anglo-American way of designation, is chosen in this paper in order to facilitate comparing of results with those published in prominent scientific journals. According to notation from [11] analyzed tones would be designated as follows: sub-contra octave (A0 as  ${}_2A$ ), contra-octave (E1 as  ${}_1E$ , A1 as  ${}_1A$ ) and great octave (E2 as  $E$ , A2 as  $A$ ).

In this paper we analyzed inharmonicity of contra octave tones generated at upright piano "August Förster", serial no. 198145, manufactured in Czechoslovakia in 1970. Coefficients of inharmonicity for all semitones from contra octave are determined and their mean value, which represents inharmonicity of the instrument, is calculated. Besides, mean aliquote frequency error (MAFE) and mean aliquote cent error (MACE) which appears as a consequence for all two-tones generated by tone C1 and other tones in this octave is also calculated.

The paper is organized as follows: Section II describes inharmonicity of instruments with strings. Section III shows an algorithm for inharmonicity coefficient estimation. Section IV shows algorithm of estimation of two-tone aliquote error. Experimental results and analysis are presented in section V. Conclusion is given in section VI. The last section lists literature used.

## II. INHARMONICITY COEFFICIENT OF MUSICAL INSTRUMENTS WITH STRINGS

When defining frequency composition of the tone, theory of music implies harmonicity, i.e. considers that harmonics (aliquotes) are integers of fundamental frequency, which is mathematically expressed as:

$$f_k = k \cdot f_0, \quad k = 1, 2, \dots \quad (1)$$

where  $f_0$  is fundamental frequency,  $k$  ordinal number of aliquote,  $f_k$  aliquote's frequency.

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Frequency displacement of aliquote from harmonics frequency position represents tone's inharmonicity. Inharmonicity is defined by means of inharmonicity coefficient  $\beta$ :

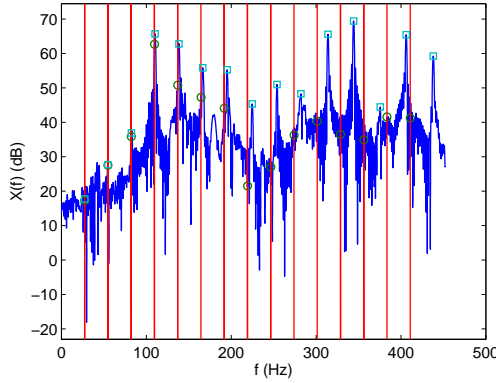
$$f_k = k \cdot f_0 \sqrt{1 + \beta \cdot k^2}, \quad k = 1, 2, \dots \quad (2)$$

Fig. 1 shows spectrum of tone played on upright piano. Vertical red lines represent frequency position of tone A0 harmonics ( $f_0 = 27.5$  Hz). These are aliquote tones A0, A1, E2, A2, C3#, E3, G3, A3, B3, C4#, D4,.... Small circles show value of amplitudal characteristic at harmonic's position, while small squares represent inharmonic's position. It is obvious that increased ordinal number of aliquote consequently increases frequency difference of harmonics and inharmonics.

Inharmonicity coefficient  $\beta$  depends of string material and can be calculated as:

$$\beta = \frac{\pi^3 \cdot Q \cdot d^4}{64 \cdot l^2 \cdot F}, \quad (3)$$

where  $Q$  represents Young's elastic modulus of material which string is made from,  $d$  strings diameter,  $l$  string's length and  $F$  tension force.



**Fig. 1.** Amplitudal characteristic of tone A0 signal.

### III. ALGORITHM FOR INHARMONICITY COEFFICIENT ESTIMATION

Literature suggests few algorithms for determination of inharmonicity coefficient  $\beta$ . In [9] an iterative algorithm is described where accuracy of determination is increasing by introducing more aliquotes, as well as by windows positioning based on previous values. Calculation represents determination of third row polynomial function coefficients which aprocsimates value of curved error. Based on polynomial's coefficients an inharmonicity coefficient is calculated. In [8] an algorithm based on introduction of interpolation function is shown. In [10] an algorithm for calculating  $\beta$ , knowing frequencies of two aliquotes without use of fundamental frequency, is proposed. Algorithm for determination of inharmonicity coefficient [10] is applied on signal  $x(n)$  and is realized in following steps:

**Step 1:** extraction of duration block  $T$ , i.e.  $N$  samples,

**Step 2:** Calculation of spectrum by applying DFT length NFFT:

$$X(i) = DFT(x(n), NFFT). \quad (1)$$

Spectrum is calculated in discrete points  $i=0, \dots, NFFT-1$ .

**Step 3:** Fundamental frequency  $f_0$  calculation by spectrum analysis and peaking of maximums,

**Step 4:** Aliquote  $f_k$  calculation, where  $k=1:N_p$ , and  $N_p$  is a number of aliquotes analyzed.

**Step 5:** Frequency difference calculation between harmonic and inharmonic of  $k$ -th aliquote:

$$e(k) = f_k - k \cdot f_0. \quad (2)$$

**Step 6:** Inharmonicity coefficient calculation:

$$\beta = \frac{\left(f_k \frac{m}{k}\right)^2 - f_m^2}{k^2 f_m^2 - m^2 \left(f_k \frac{m}{k}\right)^2}, \quad (3)$$

where  $m$  and  $k$  are aliquotes and  $f_m$  i  $f_k$  corresponding aliquote's frequencies.

### IV. ALGORITHM OF ESTIMATION OF TWO-TONE ALIQUOTE DISTORTION

Two-tones represent simultaneous sounding of two tones. Theory of aliquotes [11] sais that spectral content of one tone consists of harmonics which are, in the same time, harmonics of other tones. Simultaneous sounding of more tones means spectral overlapping of corresponding aliquotes. But, due to existence of some tones inharmonicity, aliquotes get untuned. Discrepancy of one tone's aliquotes related to other tone's corresponding aliquotes, inevitably leads to distortion of reproduced two-tone. In this paper we propose the measure of two-tone aliquote distortion. Suggested algorithm for calculation of aliquote two-sound distortion consists of following steps:

**Input:** aliquotes of two tones  $f_{1,k}$  and  $f_{2,k}$  where  $k=1:N_p$ , and  $N_p$  is the number of aliquotes analyzed.

**Output:** mean aliquote frequency error (MAFE), mean aliquote cent error (MACE), instrument's inharmonicity  $\bar{\beta}_p$ .

**Step 1:** detection of tones which form two-tone,

$$f_{1,1} \Rightarrow \text{ton\_1}; f_{2,1} \Rightarrow \text{ton\_2}. \quad (4)$$

**Step 2:** Calculation of mutual aliquotes:

$$\begin{aligned} f_{1,k} &= k \cdot f_{1,1}; f_{2,l} = l \cdot f_{2,1} \\ &\Rightarrow k \cdot f_{1,1} = l \cdot f_{2,1} \\ &\Rightarrow \{(k,l)_g, k=1, \dots, N_p, l=1, \dots, N_p, g=1, \dots, G\} \end{aligned} \quad (5)$$

where  $G$  is number of pairs  $(k,l)$  which fulfill condition of equality of aliquotes.

**Step 3:** Mean aliquote frequency error:

$$MAFE = \frac{1}{G} \sum_{g=1}^G |f_{1,k} - f_{2,l}|_g \quad (6)$$

Mean aliquote cent error:

$$MACE = \frac{1}{G} \sum_{g=1}^G |ftocent(f_{1,k}, f_{1,1}) - ftocent(f_{2,l}, f_{1,1})|_g \quad (7)$$

where  $ftocent$  is transformation function of frequency axis into cent axis with normalization to  $f_{1,1}$  frequency.

**Step 4:** Inharmonicity calculation of all two-tones in contra octave:

$$\overline{MAFE} = \frac{1}{N_D} \sum_{d=1}^{N_D} MAFE_d, \quad (8)$$

$$\overline{MACE} = \frac{1}{N_D} \sum_{d=1}^{N_D} MACE_d, \quad (9)$$

where  $N_D$  is a number of two-tones analyzed.

**Step 5:** Calculation of instrument's inharmonicity in contra octave as a mean value of some tone's inharmonicity coefficients:

$$\bar{\beta}_P = \frac{1}{N_S} \sum_{S=ton-1}^{ton-N_S} \beta_S, \quad (10)$$

where  $S$  is a continuum of tones analyzed,  $N_S$  number of continuum  $S$  elements, and  $\beta_S$  inharmonicity factor of corresponding tone.

## V. EXPERIMENTAL RESULTS AND ANALYSIS

For the purpose of calculating instrument's inharmonicity coefficient in contra octave, a database of musical signals is established. Signals are treated by algorithms described in sections III and IV. Algorithm's parameters are  $T=0.66s$ ,  $NFFT=10 \cdot 2^{18}$ ,  $m=6$ ,  $k=10$ ,  $N_p=20$ ,  $N_s=13$ . Obtained results are presented in graphics and tables.

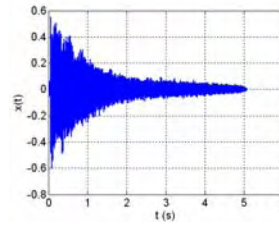
### A. Base

Base contains musical material pertaining to tones from contra octave  $S=\{C1, Cis1, D1, Dis1, E1, F1, Fis1, G1, Gis1, A1, Ais1, B1, C2\}$ . Tones are played at upright piano "August Förster" from the year 1970. Recording has been done with sampling frequency  $f_s=44.1$  kHz, and 16 bit per sample.

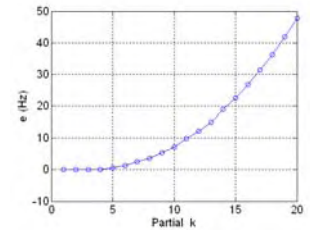
### B. Results

Referent ton's C1 time shape of signal is shown at fig. 2., Frequency difference of harmonic and inharmonic components is shown at fig. 3. Amplitudal characteristic is shown at fig. 4. Fig. 5 shows frequency positions of harmonics and inharmonics aliquotes for playing mayor third C1-E1, while positions of inharmonic aliquotes are presented at fig. 6.a (aliquote E3), fig. 6.b (aliquote E), fig. 6.c (aliquote E5). Position details of aliquote C5 with mayor third C1-C2 is shown at fig. 6.d. Values of semitones inharmonicity coefficients from con-

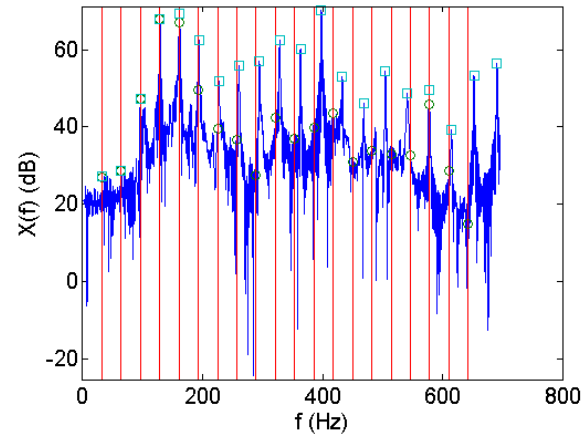
tra octave are displayed in table I. Frequency and cent values of differences of harmonious and inharmonious aliquotas are given in table II.



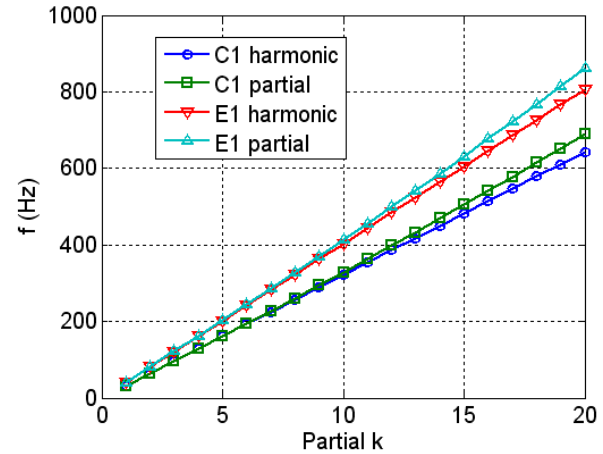
**Fig. 2.** Time shape of tone C1 referent signal.



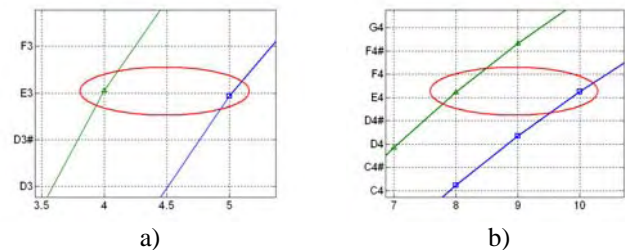
**Fig. 3.** Frequency difference of harmonious and inharmonious components (tone C1).



**Fig. 4.** Amplitudal characteristic of tone C1 signal.



**Fig. 5.** Frequency position of aliquote components (harmonics and inharmonious) when playing mayor third (C1,E1).



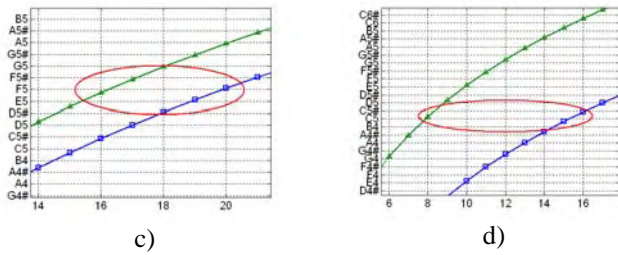


Fig. 6. Inharmonic aliquotes position details with mayor-third C1-E1: a) E3, b) E4, c) E5 and d) aliquote C5 with mayor-third C1-C2.

TABLE I  
INHARMONICITY OF SEMITONES IN CONTRA OCTAVE

Ton	$\beta_s (x10^{-4})$	Ton	$\beta_s (x10^{-4})$
C1	4.4956	G1	5.3208
Cis1	2.6631	Gis1	2.8147
D1	4.8457	A1	3.2452
Dis1	4.7512	Ais1	3.122
E1	5.088	B1	2.113
F1	4.3875	C2	2.6589
Fis1	3.399	$\bar{\beta}_p$	3.7619

TABLE II  
FREQUENTIAL AND CENT DIFFERENCE OF TWO-TONE  
HARMONIOUS AND INHARMONIOUS ALIQUOTAS IN CONTRA OCTAVE

Ton	MAFE [Hz]	MACE [cent]
C1-D1	5.0889	19.2460
C1-E1	4.3319	13.9235
C1-F1	4.4637	17.5148
C1-G1	3.0113	14.7403
C1-A1	19.8257	94.6447
C1-B1	11.5236	39.9854
C1-C2	5.1298	22.7950
	$\overline{MAFE} = 7.6250$	$\overline{MACE} = 31.8357$

### C. Results analysis

By comparing mean values of inharmonicity coefficients for semitones from contra octave (table I) with inharmonicity coefficients of piano Steinway D and Steinway C, as well as with upright piano Nordiska 1 and Straud (data from 8), we come to the following conclusion:

a) Piano Steinway D has the least inharmonicity ( $\beta_{sr}=87 \times 10^{-6}$ ),

b) baby grand piano Steinway C has less inharmonicity ( $\beta_{sr}=105.3 \times 10^{-6}$ ) than upright piano Nordiska 1 ( $\beta_{sr}=201.6 \times 10^{-6}$ ) and Straud ( $\beta_{sr}=296.22 \times 10^{-6}$ ),

c) tested "August Förster" upright piano has the largest inharmonicity in contra octave  $\bar{\beta}_p=376.19 \times 10^{-6}$  (4.32 times larger than Steinway piano).

Analyze of frequential and cent values of two-tone's harmonious and inharmonious aliquotes leads to conclusion that discrepancies in "August Förster's" contra octave have significant values in the range over tenth aliquote, where discrepancies are even larger than one semitone. We should have in

mind that these are frequencies in scope from 300 to 1200 Hz where human ear shows god sensitivity.

Our further activities will include comparative analysis of some other pianos and upright pianos inharmonicities, as well as understanding and analysis of personal impressions and experience of listeners.

## VI. CONCLUSION

An upright piano's inharmonic string's vibration effect, determined with inharmonicity coefficient, is analyzed in the paper. An algorithm for estimation of inharmonicity coefficient of strings, and whole instrument is described. In addition, an algorithm for calculating two-tone's aliquote error is proposed. That algorithm is applied to determine inharmonicity in contra octave of August Förster upright piano from 1970. Comparative analysis and presented results for piano Steinway and upright pianos Nordiska 1 and Straud proof piano's superiority comparing to upright piano's group. Upright piano August Förster showed greatest inharmonicity in contra octave which is 4.32 times larger related to grand piano. Two-tone aliquote error analysis showed that in range over tenth aliquote, discrepancies could be larger than a semitone.

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