Capacity Bounds of Lozenge Tiling Constraints

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***Abstract*—While the field of one dimensional constrained codes is mature, with theoretical as well as practical aspects of code- and decoder-design being well-established, such a theoretical treatment of two-dimensional (2D) constraints is still unavailable. A considerable research was conducted on numerous classes of 2D constraints such as hard triangle model, run-length limited constraints on the square lattice and 2D checkerboard constraints, but in spite of these efforts, these constrained systems remain largely uncharacterized mathematically with only loose bounds of capacities existing. In this paper we present a lozenge constraint on a regular triangular lattice and derive Shannon noiseless capacity bounds. To estimate capacity of lozenge tiling we make use of the bijection between the counting of lozenge tiling and the counting of boxed plane partitions.**

I. INTRODUCTION

To improve the performance of a two-dimensional (2D) recording system, a sequence of bits to be recorded onto a medium is first transformed into a pattern with properties that enable reliable reading. This transformation is called constrained coding. Constrained, because only a subset of all possible patterns is permitted to be an input to a recording channel. The readback pattern is transformed back to a binary sequence by a constrained decoder. The readback process is assumed to be errorless (noiseless). Coding for one dimensional constrained noiseless channels has been extensively studied, and myriad of codes have been developed and implemented in today's systems. It is much less known about two-dimensional constrained codes. In fact, 2D constrained coding problems are notoriously hard - only a few constraints have a closed form solution for the Shannon noiseless channel capacity, and no systematic approach exists for calculating capacity nor constrained code construction. The key novelty of the our approach is to view a 2D constraint as a colored tiling of a plane. This is a departure from the existing methods and allows using the rich theory of planar gas models from statistical mechanics and the theory of domino and lozenge tilings from combinatorics. In this paper we derive a framework for designing encoders and decoders for a triangular grain model.

Two-dimensional constraint is a restriction on a coloring (or labeling) of tiles in a regular tiling. The most famous example is a hard-hexagon constraint, which is a planar lattice gas model with nearest-neighbor exclusion used in statistical mechanics. The hard-hexagon constraint allows only those (binary) colorings in which black hexagons are isolated, i.e. have all white neighbors. A hard-triangle and hard-sqare constraints are defined similarly. A two-dimensional *runlength* (d, k) constraint is a restriction on the separation space between black tiles, so that the number of white tiles between two black tiles in any direction is at least d and at most k. Hard-hexagon, hard-triangle and hard-square constraints are $(d, k) = (1, \infty)$ runlength constraints.

In [1] we have shown that constrained coding which restricts the occurrence of such patterns greatly reduces system complexity and improves the detection performance. In our prior work [2] we considered a tiling with rectangular prototiles representing grains. More generally, collections of adjoining cells are called *polyominoes* or *animals*, objects studied in combinatorial mathematics [3]. The recording medium can now be modeled as a tiling of a plane with a given set of polyominoes and with an appropriately chosen probability distribution. This can be simplified by restricting the possible shapes of grains. Although algorithms for generating domino tilings of planes are known [4]–[6], tiling of planes with polyominoes whose occurrences are governed by a probability distribution is a non-trivial problem (*e.g.* [3], [7]). The *Shannon noiseless capacity* of a 2D constraint is defined as an asymptotic growth rate of a number of distinct (reconstructible) patterns permitted by the constraint. The main idea is to estimate the number of permissible tilings, which will lead to upper bounds on Shannon noiseless capacity of 2D constrained channels, and determine the achievable rates of the constrained codes.

The rest of the paper is organized as follows. In Section II we first introduce a lozenge constraint and establish the connection between lozenge tilings and boxed plane partition. Then we proceed to determine the capacity bounds by splitting the problem into tiling and coloring. Section IV concludes the paper and puts our work in the perspective with efforts in combinatorics.

II. LOZENGE CODES

Consider a regular tiling of a plane. A (d, k) segregation constraint is a tiling that limits the number of neighboring tiles (neighborhood size) of same color to be no less than $d + 1$ and no more than $k + 1$. A 2D bit pattern is said to satisfy a *no isolated bit* (NIB) constraint if every bit has at least one bit of the same polarity adjacent to it. Thus the NIB is a $(d, k) = (1, \infty)$ segregation constraint.

Consider a hexagon $\mathcal{H}$ with sides of lengths a, b, c, a, b, c and angles of $2\pi/3$ subdivided into equilateral triangles of unit side by lines parallel to the hexagon sides. We will henceforth refer to such equiangular triangularized hexagons as (a, b, c)

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hexagons. Figure 1(a) shows such a $(2,3,4)$ hexagon. As mentioned earlier, two triangles are neighbors if they share a side.

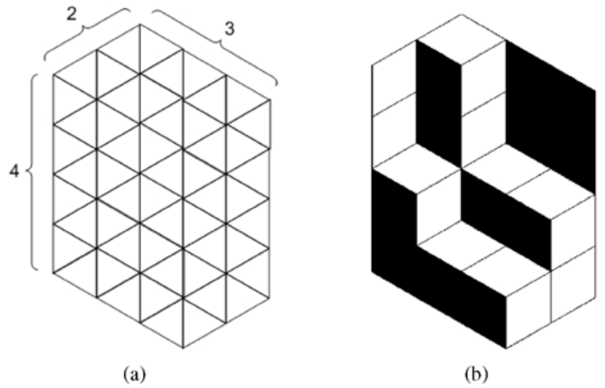

Fig. 1. Lozenge tilings in a triangular lattice: (a) A $(2,3,4)$ hexagon $\mathcal{H}$ embedded in a triangular lattice. (b) Colored tiling of a (2,3,4) hexagon.

We are interested in coloring the triangles using M different colors in such a way that no isolated triangle is colored differently than its neighbors. We focus mainly on the $M = 2$ case. We require a segregation of at least two neighboring triangles of the same color. Fig. 1(b) shows one such coloring of the lozenge-tiled $(2,3,4)$ hexagon shown in Fig. 1(a).

Two neighboring triangles form a rhombus with side of unit length and internal angles $\frac{\pi}{3}$ and $\frac{2\pi}{3}$. Such a rhombus is known as a *lozenge*. A lozenge created in this way may have three different orientations. Therefore, for any triangle there must be at least one neigboring triangle with the same color, i.e. $(d,k) = (1,\infty)$. More specifically, we are interested in the constraint which allows independent coloring of different lozenges. This means that the neighborhood size is at least two, and is even.

Now the NIB coloring can be separated into two steps as illustrated in Fig. 2. First, the triangular grid is tiled with (uncolored) lozenges, and then the lozenges are colored independently.

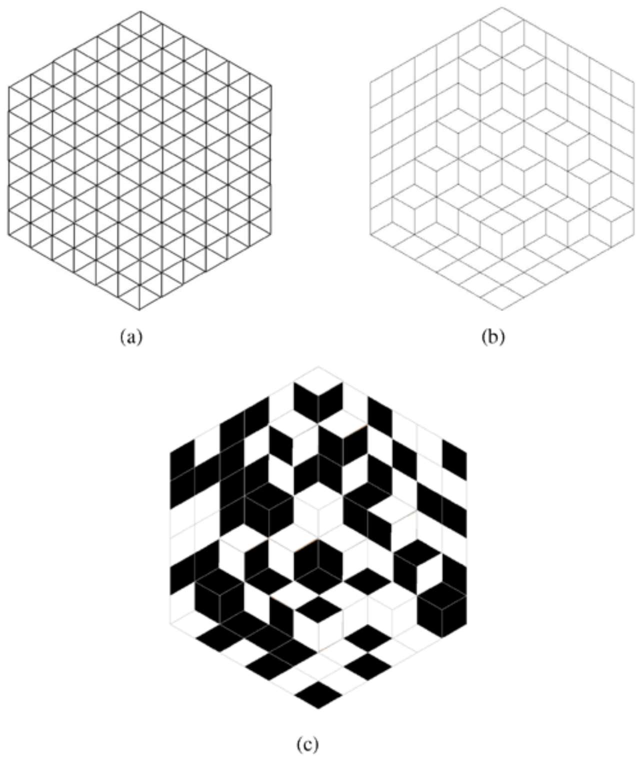

Fig. 2. Colored lozenge tilings in a triangular lattice: (a) A hexagon with side-length of 6 embedded in a triangular lattice. (b) An uncolored lozenge tiling of the hexagon. (c) A colored lozenge tiling of the hexagon.

III. Bounds on Capacity

For a fixed shape on the lattice where the information is stored (a hexagon in our example), one would like to find the number of distinct colored patterns as a function of the area, and/or its behavior when the size grows to infinity. For an equilateral hexagon with side n on the triangular lattice, the Shannon noiseless capacity is

$$C = \lim_{n\to\infty} \frac{\log_2 M(n,n,n)}{6n^2}$$

where $M(n,n,n)$ is the number of distinct two-colored patterns ($6n^2$ is the number of triangles in the hexagon.) Similarily, if $N(n,n,n)$ denotes the number of uncolored patterns (tilings), the asymptotic growth rate of $N(n,n,n)$ is referred as *tiling capacity*.

In contrast to existing approaches, our method is inspired by the theory of domino tilings of lattices. In particular, we are interested in calculating the number of *colored domino tilings* of a given lattice. Observe that this will yield bounds on the capacity of the NIB constraint because the problem of estimating the capacity $C_{(1,\infty)}$ of the NIB constraint for the triangular lattice model, can be divided into two simpler independent problems: tiling of a hexagon with lozenges and the coloring the lozenge tilings.

Given that information storage capacities of tiling and coloring of tiling are C_T and C_C, respectively, the capacity of the constraint can be bounded from below by the sum of C_T and C_C. That is,

$$C_{(1,\infty)} \geq C_T + C_C \tag{1}$$

A. A Method Based on Boxed Plane Partitions

A *plane partition* π is a collection of non-negative integers $\pi_{x,y}$ indexed by non-negative integers x, y such that (a) only finite number of $\pi_{x,y}$ are non-zero and, (b) $\forall x, y\ \pi_{x,y} \leq \pi_{x,y+1}$ and $\pi_{x,y} \leq \pi_{x+1,y}$. The plane partition is said to *fit inside* a box of dimension $a \times b \times c$ if there exist integers a, b, c such that $\pi_{x,y} \leq c$ for all x, y and $\pi_{x,y} = 0$ for all $x > a, y > b$. Such partitions are called *boxed plane partitions*. A more intuitive way of visualizing the boxed plane partitions is by constructing the *Young's solid diagram* corresponding to a boxed partition π. For instance, consider the following partition boxed within a box of dimension $2 \times 3 \times 4$.

$$\pi = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \end{pmatrix}$$

The dimension of the matrix is 2×3 with all entries ≤ 4. To build its corresponding Young's solid diagram, we first consider a box of dimension $2 \times 3 \times 4$. To one of the vertices, we assign the Cartesian co-ordinate $(0, 0, 0)$. To the opposite vertex we assign the co-ordinate $(2, 3, 4)$. For each $x \leq 2, y \leq 3$, we start at $(x-1, y-1, 0)$ and stack $\pi_{x,y}$ cubes of unit side length. This construction will fit inside the box.

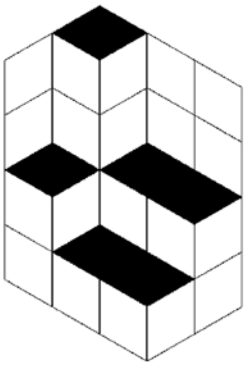

Fig. 3. Tiling corresponding to the boxed plane partition π.

The tiling given in Fig. 3 corresponds to the boxed plane partition π given above. The Young's solid diagram construction is also apparent if the tiled $(2, 3, 4)$ hexagon is visualized as a $2 \times 3 \times 4$ box. For ease of visualization, the top face of the topmost box is colored black. The relationship between lozenge tiling and boxed plane partition is given in [8].

Lemma 1: The number of boxed plane partitions fitting inside a $a \times b \times c$ box is equal to the number of lozenge tilings of an (a, b, c) hexagon.

Proof: The proof is based on MacMahon [9] formula.

B. Estimation of C_T

By Lemma 1, the number of lozenge tiling of any (a, b, c) hexagon is equal to the number of boxed partitions fitting inside an $a \times b \times c$ box. MacMahon [9] found the number of such partitions, $N_{(a,b,c)}$, to be $N_{(a,b,c)} = \prod_{i=1}^{a} (c+i)_b/(i)_b$ where $(i)_n := i(i+1)(i+2)\ldots(i+n-1)$ is the *rising factorial*. For an equilateral hexagon, we were able to find the capacity of tiling (without coloring) in a closed form. It is given by Theorem 1.

Theorem 1: The capacity of the lozenge tiling for an equilateral hexagon is $C_T = \frac{3}{4}\log_2 3 - 1$.

Proof: The proof follows from Lemma 1, but is omitted due to space limitations.

Note that closed forms solutions for the capacity such as one given by the above theorem are quite rare in problems involving 2D constraints.

C. Estimation of C_C

The capacity of the coloring problem, C_C, can be estimated by counting the number of ways the lozenges constituting a lozenge tiling can be colored. Given a lozenge tiling of an (n, n, n) hexagon, $\mathcal{H}$, the coloring of the lozenges cannot be done arbitrarily. To explain this, we consider a colored lozenge tiling of the $(2, 1, 1)$ hexagon which is shown in Figure 4. Figure 4(a) shows the colored lozenge tiling without any boundaries. In the absence of the tiling information, the decoder would not be able to decode this tiling correctly. Figures 4(b) and 4(c) show two possible lozenge tilings this can be decoded to. This means that in the absence of knowledge of the tiling pattern, the colored tiling in Figure 4(a) is not uniquely decodable. In specific, the two tilings of the $(1, 1, 1)$ hexagon formed at the bottom of the larger hexagon (marked with bold edges) cannot be identified uniquely.

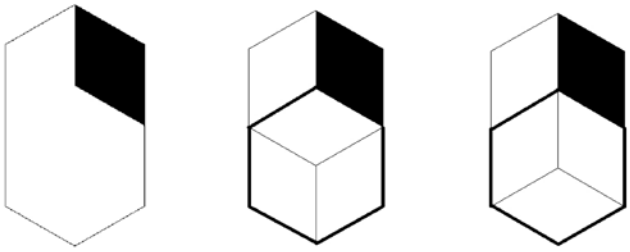

Fig. 4. Figure (a) shows a colored lozenge tiling for a $(2, 1, 1,)$ hexagon. In the absence of tiling information, the colored tiling in Figure (a) can be decoded to either of the colored tilings shown in Figures (b) and (c)

We denote the $(1, 1, 1)$ hexagon formed in Figures 4(b) and 4(c) as type-1 and type-2 hexagon, respectively. To distinguish the two hexagons, we apply the following rule: Whenever a type-1 hexagon is encountered, the vertical lozenges constituting it are colored differently. By using this coloring scheme, no information is stored in the vertical lozenges of the type-1 hexagon. This constraint reduces the number of ways of coloring for each tiling. This reduction depends on the number of type-1 hexagons formed in a tiling. The capacity can be bounded by calculating the number of type-1 hexagons formed for every tiling.

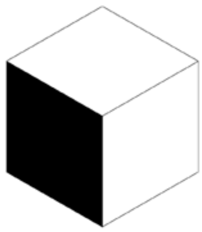

Fig. 5. Figure shows the coloring scheme adopted for type-1 hexagon.

The capacity of the coloring problem, C_C, can be estimated by counting the number of ways the lozenges constituting a lozenge tiling can be colored. Given a lozenge tiling of an (n, n, n) hexagon, $\mathcal{H}$, the coloring of the lozenges cannot be done arbitrarily.

Theorem 2: For the coloring of a lozenge tiling of an (n, n, n) hexagon $\mathcal{H}$, the capacity $C_C \geq \frac{1}{3}$.

Proof: The proof is based on the equivalence between tilings and boxed plane partitions and Lemma 1 It is given in Appendix A.

Note that the above bound on C_C may be strengthen at the expense of much harder combinatorial argument considering colors of tiles.

IV. Conclusion

The use of colored tilings to estimate capacity is attractive due to the fact that the theory of counting domino tilings is well-explored [8], [10]. Early work in counting domino tilings includes the work of Kasteleyn [4], who calculated the number of domino tilings of a square lattice. Another work is that of MacMahon [9] in which the number of lozenge tilings of a hexagon embedded in a triangular lattice was calculated. Desreux and Remila [6] gave optimal algorithms for the generation of domino tilings and lozenge tilings. We have exploited these results to establish bounds on constrained codes on the regular triangular lattice.

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Appendix A

Proof of Theorem 2: MacMahon formula [9] gives the number $N(a,b,c)$ of restricted plane partitions in the form

$$N(a,b,c)=\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}$$

It can be rewritten as

$$N(a,b,c)=\prod_{i=1}^{a}\frac{(c+i)_b}{(i)_b}$$

where $(i)_n := i(i+1)(i+2)...(i+n-1)$ is the *rising factorial.*

Since $(i)_n = (i+n-1)!/(a-1)!$, we have

$$\prod_{i=1}^{a}\frac{(c+i)_b}{(i)_b}=\prod_{i=1}^{a}\frac{(c+i+b-1)!(i-1)!}{(c+i-1)!(i+b-1)!}$$

Each product of factoriels in A is of the form $\prod_{i=1}^{a}(d+i-1)!$ and can be written as

$$\prod_{i=1}^{a}(d+i-1)!=\frac{\prod_{i=1}^{d+a-1}i!}{\prod_{i=1}^{d-1}i!}$$

The superfactorials $\prod_{i=1}^{a} i!$ in A can be expressed as $\prod_{i=1}^{n-1} i! = G(n+1)$. Therefore we obtain

$$N(a,b,c)=\prod_{i=1}^{a}\frac{G(a+b+c+1)G(a+1)G(b+1)G(c+1)}{G(a+b+1)G(a+c+1)G(b+c+1)} \tag{2}$$

where in Eq. 2 G denotes the Barnes G-Function [11] defined by

$$G(d+1)=(2\pi)^{\frac{d}{2}}e^{-\frac{1}{2}\left(d(d+1)+\gamma d^2\right)}\cdot \cdot\prod_{i=1}^{+\infty}\left(\left(1+\frac{d}{i}\right)^{i}e^{-d+d^2/(2i)}\right)$$

For a regular hexagon with $a=b=c=n$ we have

$$N(n,n,n)=\prod_{i=1}^{a}\frac{G(3n+1)\left(G(n+1)\right)^3}{G(2n+1)^3} \tag{3}$$

Using the asymptotic of $G(d+1)$

$$\ln G(d+1)\tilde{\ } z^2(\frac{1}{2}\ln z-\frac{3}{4})+\frac{1}{2}\ln(2\pi)z- -\frac{1}{12}\ln z+\xi'(-1)O(\frac{1}{2})$$

we obtain

$$\begin{aligned}\ln N(n,n,n) &\sim \ln G(3n+1)+3\ln G(n+1)- \\ &\quad -3\ln G(2n+1)\\ &\sim n^2\left(\frac{9}{2}\ln 3-6\ln 2\right)+\\ &\quad +n\ln 2\pi-\frac{1}{12}\ln n-\frac{1}{12}\ln\frac{3}{2}\end{aligned}$$

By changing the base of the logarithm we obtain

$$\log_2 N(n,n,n)\sim n^2\left(\frac{9}{2}\log_2 3-6\right) \tag{4}$$

The area of the hexagon with sides a,b and c is $A(a,b,c) = 2(ab+ac+bc)\frac{\sqrt{3}}{4}$. For a regular hexagon $A(n,n,n)=\frac{2\sqrt{3}}{2}n^2$, so that finally the density is

$$\begin{aligned}D &= \lim_{n\to\infty}\frac{\log_2 N(n,n,n)}{A(n,n,n)}\\ &= \lim_{n\to\infty}\frac{n^2\left(\frac{9}{2}\log_2 3-6\right)}{\frac{2\sqrt{3}}{2}n^2}\\ &= \sqrt{3}(\log_2 3-\frac{4}{3})\end{aligned}$$

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