Analysis of Class Matrices for Complex Hadamard Transform

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***Abstract* – An analysis of class matrices for complex Hadamard transformation (CHT) is presented. The concept of a family of CHT's is introduced and such a family is proposed to represent discrete signals and systems. Several properties of this family are outlined. The generation of the transformation matrices commences from the basic Walsh-Hadamard transformation matrix.**

***Keywords* – digital signal processing, Complex Hadamard Transform, orthogonal transforms.**

I. Introduction

In this article, a class matrices for discrete orthogonal transformations with elements which are integer-valued complex numbers and may be considered as systems of complex Walsh functions are introduced. These transforms may be useful in applications where the need for complex-valued discrete orthogonal transforms arises, such as digital signal processing, spectral analysis, pattern recognition, digital coding, computational mathematic and etc. These systems of functions and transformations are called complex Hadamard transforms (CHT's) and are confined to four complex values (±1 and ±j). Dimensionality reduction in computation is a major signal processing application. Stated simply, these transform coefficients that are small may be excluded from processing operations, such as filtering, without much loss in processing accuracy. In the literature, there exists another transformation based on four-valued complex Walsh functions, called the "complex BIFORE transform" [1], [2]. For real-valued input data, the complex BIFORE transform reduces to a BIFORE or Hadamard transform whose bases are Walsh functions. The common definition of the complex BIFORE transform is based on a recursive formula defining one class of complex Hadamard matrices that involves diagonalization of higher order matrices and multiple Kronecker products. The unified complex Hadamard transforms (UCHT's) have recently been considered as a tool in spectral approach to logic design [3], [4]. Like its predecessors, the UCHT's show similar properties and characteristics [5]. The idea of using complex-valued rather than integer-valued transformation matrices for spectral processing of Boolean functions by using Perkowski linearly independent logic is considered for the first time in article [6]. By increasing still further the number of possible different entries in the transformation matrices with complex numbers, one can expect the reduction of their spectral representation, especially if both the original functions and their spectra are presented in the form of some kind of decision diagrams

In particular, the Walsh–Hadamard transform is one of many UCHT matrices introduced here. Some of the UCHT matrices have a unique half-spectrum property (HSP). There are general fast algorithms from the representation of transform matrices in the form of layered Kronecker matrices. In addition, constant-geometry fast algorithms with in-place architecture are also available for the new transforms. The complex BIFORE transform instead has only fast transform without constant geometry algorithm. The existence of constant-geometry fast butterflies is suitable for efficient very large-scale integration (VLSI) implementation. The introduced UCHT's may be used for various applications, where the Walsh-Hadamard transform has already been used [7], [8], [9], [10], [11]. Generally, the UCHT's may be classified as the integer-valued and complex integer valued transforms. The integer-valued and complex integer-valued matrices have elements confined to two (±1) and four complex numbers (±1 and ±j), respectively. Comparing the complex integer valued UCHT's between themselves, those that possess HSP will be advantageous as they require half of the spectral coefficients for analysis. However, it should be pointed out that if the functional data are real numbers, the existence of the HSP in complex integer valued UCHT's has no additional storage advantage compared to the integer-valued counterparts (e.g., Walsh-Hadamard transform) [4]. But, the complex integer-valued transforms are suitable for problems with complex-valued functions and for such functions, the CHT's with half spectrum property is the most compact representation. Some UCHT's are simply systems of complex Walsh functions while others become q-valued Chrestenson functions for q = 2 or 4 [12], [13]. It is then obvious that the UCHT's can be used in different applications of complex Walsh functions and Chrestenson functions in processing of multiple-valued functions, especially for the case of four-valued functions.

From the Complex Hadamard Transform (CHT), several complex decisions diagrams are derived and analysis of more general CHT properties for 1D and 2D signals are investigated [10], [11].

In this paper, the concept of a family of CHT's is introduced and such a family is proposed to represent discrete signals and systems. Several properties of this family are

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outlined. The generation of the transformation matrices commences from the basic Walsh-Hadamard transformation matrix. All members of UCHT's may be produced by the defined direct matrix operator and recursively generated to higher dimension matrices by a single Kronecker product. It must be noted that although the basis functions in the definition that generates all UCHT matrices are discrete Walsh functions, each member of the newly defined UCHT fulfills requirements of complex Hadamard matrices; there are altogether 64 such different matrices that are introduced in this section, all of which are generated by one unifying formula.

II. MATHEMATICAL DESCRIPTION

In the definitions of existing discrete orthogonal transforms, the elements of transformation matrices normally consist of discrete values of +1 and -1; or generalizations that permit values of $e^{2\pi mj/q}$ for a prime q, $j=\sqrt{-1}$; which leads to a complete orthonormal system known as the Chrestenson system [12], [133].

The transformation matrices are defined by a set of basis discrete valued functions. To ensure that no information is lost in the resulting spectrum, orthogonality in the transformation matrix is essential. This requires zero correlation between pairs of different basis functions. In general, [H] is an orthogonal NxN matrix with real entries, when:

$$[H]^T = [H]^{-1} \quad . \tag{1}$$

Then, the following relation is fulfilled:

$$[H].[H]^T = [H]^T.[H] = N.[I] \quad , \tag{2}$$

where [I] is the identity matrix.

In the complex domain [H] is orthogonal if:

$$[H].[H]^* = [H]^*.[H] = N.[I] \quad , \tag{3}$$

where $[H]^*$ is the represents the complex conjugate transpose of [H], $\left|det[H]\right| = N^{1/2N}$ and [H] is said to be a CHT. The resulting matrix [H] can be easily used as a binary, ternary, or quaternary transform as any two, three, or all four elements in the transformation matrix can be used for coding of two-, three-, or four-valued logic functions respectively. In addition, with an appropriate coding of the original function, the UCHT may be used as a multiple-valued transform.

All UCHT matrices of size 2x2 can be separated into two groups of 32 basic matrices dependent on the existence of the HSP [4]. These transformation matrices are listed in Table 1.

The transformation core matrix for any UCHT is defined as:

$$[H]_1^{[r]} = [W]_1 \circ \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \circ \begin{bmatrix} 1 & r_3 \end{bmatrix} , \tag{4}$$

where $[W]_1$ is the Walsh–Hadamard transform matrix of order 2 and matrix operator $\circ$ is defined as:

$$[A \circ B] = \{a(k,l) \quad b(l)\}$$

and

$$[A \circ C] = \{a(k,l) \quad c(k)\}. \tag{5}$$

Here $[A]$ is an r x c matrix, such as $[A] = \{a(k,l)\}$, where $0 \le k \le r$, $0 \le l \le c$. $a(k,l)$ is the current element of matrix $[A]$ at row k and column l; $[B]=\{b(l)\}$ is 1 x c row matrix and $[C]=\{c(k)\}$ is an r x 1 column matrix.

As is shown in [4] the following properties of $\circ$ are derived:

$$\begin{aligned}
&(A \circ B)\otimes^n = (A\otimes^n)\circ(B\otimes^n) \\
&(AB^T)\circ C = (A\circ C)B^T \\
&(A\circ B_1)\circ B_2 = A\circ(B_1\circ B_2) \\
&(A\circ B\circ C)^T = (A\circ B)^T\circ C^T \\
&\overline{(A\circ B\circ C)} = \overline{(A\circ B)}\circ\overline{C} = \overline{A}\circ\overline{B}\circ\overline{C}
\end{aligned} \quad . \tag{6}$$

TABL.1

$\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$	$\begin{bmatrix}j & j\\ 1 & -1\end{bmatrix}$	$\begin{bmatrix}-1 & -1\\ 1 & -1\end{bmatrix}$	$\begin{bmatrix}-j & -j\\ 1 & -1\end{bmatrix}$
$\begin{bmatrix}1 & j\\ 1 & -j\end{bmatrix}$	$\begin{bmatrix}j & -1\\ 1 & -j\end{bmatrix}$	$\begin{bmatrix}-1 & -j\\ 1 & -j\end{bmatrix}$	$\begin{bmatrix}-j & 1\\ 1 & -j\end{bmatrix}$
$\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$	$\begin{bmatrix}1 & -j\\ 1 & 1\end{bmatrix}$	$\begin{bmatrix}-1 & 1\\ 1 & 1\end{bmatrix}$	$\begin{bmatrix}-j & j\\ 1 & 1\end{bmatrix}$
$\begin{bmatrix}1 & -j\\ 1 & j\end{bmatrix}$	$\begin{bmatrix}j & 1\\ 1 & j\end{bmatrix}$	$\begin{bmatrix}-1 & j\\ 1 & j\end{bmatrix}$	$\begin{bmatrix}-j & -1\\ 1 & j\end{bmatrix}$
$\begin{bmatrix}1 & 1\\ j & -j\end{bmatrix}$	$\begin{bmatrix}j & j\\ j & -j\end{bmatrix}$	$\begin{bmatrix}-1 & -1\\ j & -j\end{bmatrix}$	$\begin{bmatrix}-j & -j\\ j & -j\end{bmatrix}$
$\begin{bmatrix}1 & j\\ j & 1\end{bmatrix}$	$\begin{bmatrix}j & -1\\ j & 1\end{bmatrix}$	$\begin{bmatrix}-1 & -j\\ j & 1\end{bmatrix}$	$\begin{bmatrix}-j & 1\\ j & 1\end{bmatrix}$
$\begin{bmatrix}1 & -1\\ j & j\end{bmatrix}$	$\begin{bmatrix}j & -j\\ j & j\end{bmatrix}$	$\begin{bmatrix}-1 & 1\\ j & j\end{bmatrix}$	$\begin{bmatrix}-j & j\\ j & j\end{bmatrix}$
$\begin{bmatrix}1 & -j\\ j & -1\end{bmatrix}$	$\begin{bmatrix}j & 1\\ j & -1\end{bmatrix}$	$\begin{bmatrix}-1 & j\\ j & -1\end{bmatrix}$	$\begin{bmatrix}-j & -1\\ j & -1\end{bmatrix}$
$\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$	$\begin{bmatrix}j & j\\ -1 & 1\end{bmatrix}$	$\begin{bmatrix}-1 & -1\\ -1 & 1\end{bmatrix}$	$\begin{bmatrix}-j & -j\\ -1 & 1\end{bmatrix}$
$\begin{bmatrix}1 & j\\ -1 & j\end{bmatrix}$	$\begin{bmatrix}j & -1\\ -1 & j\end{bmatrix}$	$\begin{bmatrix}-1 & -j\\ -1 & j\end{bmatrix}$	$\begin{bmatrix}-j & 1\\ -1 & j\end{bmatrix}$
$\begin{bmatrix}1 & -1\\ -1 & -1\end{bmatrix}$	$\begin{bmatrix}j & -j\\ -1 & -1\end{bmatrix}$	$\begin{bmatrix}-1 & 1\\ -1 & -1\end{bmatrix}$	$\begin{bmatrix}-j & -j\\ -1 & -1\end{bmatrix}$
$\begin{bmatrix}1 & -j\\ -1 & -j\end{bmatrix}$	$\begin{bmatrix}j & 1\\ -1 & -j\end{bmatrix}$	$\begin{bmatrix}-1 & j\\ -1 & -j\end{bmatrix}$	$\begin{bmatrix}-j & -1\\ -1 & -j\end{bmatrix}$
$\begin{bmatrix}1 & 1\\ -j & j\end{bmatrix}$	$\begin{bmatrix}j & j\\ -j & j\end{bmatrix}$	$\begin{bmatrix}-1 & -1\\ -j & j\end{bmatrix}$	$\begin{bmatrix}-j & -j\\ -j & j\end{bmatrix}$
$\begin{bmatrix}1 & j\\ -j & -1\end{bmatrix}$	$\begin{bmatrix}j & -1\\ -j & -1\end{bmatrix}$	$\begin{bmatrix}-1 & -j\\ -j & -1\end{bmatrix}$	$\begin{bmatrix}-j & 1\\ -j & -1\end{bmatrix}$
$\begin{bmatrix}1 & -1\\ -j & -j\end{bmatrix}$	$\begin{bmatrix}j & -j\\ -j & -j\end{bmatrix}$	$\begin{bmatrix}-1 & 1\\ -j & -j\end{bmatrix}$	$\begin{bmatrix}-j & j\\ -j & -j\end{bmatrix}$
$\begin{bmatrix}1 & -j\\ -j & 1\end{bmatrix}$	$\begin{bmatrix}j & 1\\ -j & 1\end{bmatrix}$	$\begin{bmatrix}-1 & j\\ -j & 1\end{bmatrix}$	$\begin{bmatrix}-j & -j\\ -j & 1\end{bmatrix}$

In Eqs. (6) the matrices $[B_1]=\{b_1(l)\}$ and $[B_2]=\{b_2(l)\}$ are $1 \times c$ row matrices and $\otimes^n$ denotes right-hand side Kronecker product applied n times.

The basis complex Hadamard matrices of order 2^n (n>2) can be received as the Kronecker product of a number of identical "core" matrices of order 2^{n-1} in the following way:

$$\left[CH_{2^n}\right]=\begin{bmatrix}[CH_{2^{n-1}}] & [CH_{2^{n-1}}]\\ [CH_{2^{n-1}}] & -[CH_{2^{n-1}}]\end{bmatrix}. \qquad (7)$$

Using the basic forward one-dimensional complex Hadamard transform for n=2 from the input signal vector $\vec{X}=\left[x_1,x_2,x_3,........,x_N\right]$, the output spectral vector - $\vec{Y}=\left[y_1,y_2,y_3,........,y_N\right]$ is received by the equations [10]:

$$\left|\begin{array}{l}\vec{Y}=\ [CH_N]\,\vec{X}\\ \vec{X}=\dfrac{1}{N}[CH_N]\,\vec{Y}\end{array}\right. \quad \text{for:} \quad \left|\begin{array}{l}\vec{Y}=\{y(u)/u=\overline{0,N-1}\}\\ \vec{X}=\{x(v)/v=\overline{0,N-1}\}\end{array}\right. . \qquad (8)$$

From the above equations the following mathematical properties can be established:

$$\left|\det\left[CH_N\right]\right|^2=N^N \qquad (9)$$

$$\left[CH_N\right]\left[CH_N\right]^*=N\left[I\right] \qquad (10)$$

$$\left[CH_N\right]^{-1}=\frac{1}{N}\left[CH_N\right]^* \qquad (11)$$

$$\left[CH_N\right]\left[CH_N\right]^t=\left[CH_N\right]^t.\left[CH_N\right]=N\left[I\right] \qquad (12)$$

The common results, obtained from the one dimensional Complex Hadamard Transform can be generalized for two-dimensional Complex Hadamard Transform. In this case the 2D signals (images) can be represented by the input matrix [X] with the size NxN. The result is a spatial spectrum matrix [Y] with the same size. The corresponding equations for the forward and the inverse 2D CHT are:

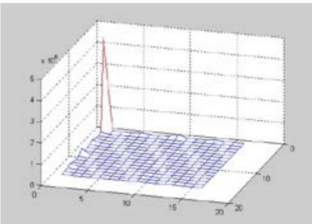
$$\left|\begin{array}{l}\left[Y\right]=\ [CH_N]\left[X\right][CH_N]\\ \left[X\right]=\dfrac{1}{N^2}[CH_N]\left[Y\right][CH_N]\end{array}\right. \qquad (13)$$

The symmetry of CHT coefficients allows 2D CHT to be accomplished in two steps. The first one is 1D CHT for every row the image and the second one is 1D CHT for the columns. This difference of transformation makes easier the calculations and the symmetry guarantees that the correlations between image elements in horizontal and vertical direction will influence in the same way the determination of transformed elements. The same considerations can be made for two steps calculation of the inverse 2D CHT.

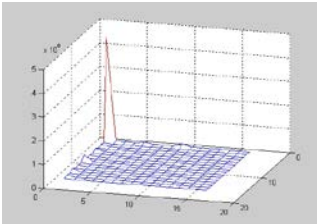
III. Experimental Results

For the analyses of spectral distribution between the coefficients of 2D CHT, constructed with the different base matrices of order 2, a test image "LENNA", shown in Fig.1, with size 512x512 and 256 gray levels is used. This image is transformed by the 2D CHT with kernel 16x16. By this way the input image is divided on 1024 sub-images with size 16x16 and is calculated by MATLAB 6.5 program. In Fig.2a and Fig. 2b the averaged amplitude frequency spectrums of all sub-blocks for two-dimensional Complex Hadamard Transform and two dimensional real Hadamard Transform respectively, are shown. On Fig. 2c the averaged phase frequency spectrum calculated for all sub-blocks, for two-dimensional Complex Hadamard Transform, is shown.

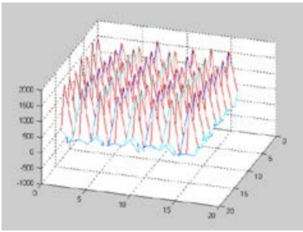
Fig.1. Test image "LENNA" (512x512 pixels and 256 gray levels).



a) 2D CHT amplitude spectrum;



b) 2D HT amplitude spectrum;



c) 2D CHT phase spectrum;

Fig. 2. Averaged amplitude and phase spectrums of image "LENNA"

IV. Conclusion

A class of Complex Hadamard matrices is presented.

The CHT is based on the mapping of 4-valued integers into the unit circle of the complex plane with elements strictly in the set {1;-1; j;-j}. Under the various permutations of the integers, there exist some conditions which will lead to the transform being mapped to and being orthogonal in the complex domain. This has been identified as the family of UCHT's, as the mapping of the multiple-valued transforms into the complex domain will result in square basis matrices which satisfy the Hadamard's determinant equation in the complex domain. Intuitively, Walsh–Hadamard being an integer-valued transform is merely a special case of the UCHT's.

Another advantage of the presented transform is the existence of not only fast algorithms based on layered Kronecker products that can be represented by a series of strand matrices (which is similar to the complex BIFORE transform), but also a constant geometry fast algorithm that is well suited to VLSI hardware implementation. In such architecture, only one butterfly stage has to be implemented and the processed data can be fed back to the input to be processed by the same circuitry.

The general principles of complex matrices construction of high order for 1D and 2D transforms are given. The basic properties of CHT are discussed. The obtained amplitude spectrums for CHT and HT are practically identical and show that both can be used in similar applications.

The presented Complex Hadamard Transform can be used in digital signal processing for spectral analysis, pattern recognition, digital watermarking, coding and transmission of one-dimensional and two-dimensional signals.

Signal parameters in many DSP applications are estimated using the Fourier power spectrum. However, computing the Fourier transform is relatively complicated and there are applications for which it is important to achieve hardware savings, even at the expense of some loss in parameter estimation accuracy, as is the case in satellite radar altimetry. The Walsh–Hadamard transform is used for such an application. Also, it is well known from the literature that the fast Walsh–Hadamard transform can be efficiently used for the calculation of the DFT for implementing adaptive filters and for DFT spectrum filter realizations. The usual frequency-domain FIR filtering problem can be easily converted into a Walsh frequencydomain filtering problem, and similar structure results in a possible alternative for infinite-impulse response filter implementations. An efficient method for implementation of a class of isotropic quadratic filters using the Walsh–Hadamard transform was also proposed. Advantages of the 2-D Walsh–Hadamard transform, also known as S or sequential transform, in lossless image compression are well known. An integrated-circuit chip implementing 2-D Walsh–Hadamard transform has been implemented for commercial applications by Philips Corporation. As the Walsh–Hadamard transform is one of the UCHT's, it is thus believed that the important properties of the UCHT's presented in this article may also be of interest to researchers working in the above-mentioned areas where the standard Walsh–Hadamard matrices had been applied.

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