Lambert W Function Application to Time-Delay Automatic Control Systems

Radmila Gerov[1] and Zoran Jovanović[2]

Abstract – **The paper presents the design of the P (proportional) controller for integral time-delay systems by employing a pole placement method. By determining the set-point conditions on the selection of the dominant poles and the amplification coefficient of the controllers, systems stability in the absence of perturbations is guaranteed.**

Keywords – **P controller, Integral Time Delay Systems, Lambert W function, Pole placement.**

I. Introduction

There are numerous reasons for a delay in dynamic systems: the delay of the material, signal, sensor responses… The systems functioning are not only affected by the current values of their variables (states), but also by their previous values, which is why they are difficult to maintain [1].

During system design, the quality of the transitory mode and system functioning in a static mode need to be taken into account. The Lyapunov-Krasovskii approach is usually used for analyzing the stability of time-delay systems (TDS) [2], but also other traditional approaches, which are extremely difficult to apply, can be used: the Routh-Hurwitz criterion, the Nyquist plot [3], as well as numerous other methods common in literature.

The control of TDS, especially in industry, is carried out by a proportional–integral–derivative controller (PID) [4], proportional-integral controller (PI) [5], the Smith predictor [6] and its modified versions... The desired systems functioning can be realized only with carefully selected controller parameters, where various techniques are used; a pole placement [7], parameter optimization, optimal criteria, as well as the application of Lambert W function (LWF) [8].

By using the pole placement approach, the paper analyzed the maintenance of the integral first-order time-delay system (FOTDS) and the astatic first-order time-delay system with a proportional controller (P). The sufficient conditions for the selection of the amplification of the controllers and dominant poles were established by using LWF [9]. The response received by simulation was obtained for different values of the desired poles, and it was shown that the sufficient conditions were accurately determined. LambertW_DDE Toolbox [10] was used for calculations.

The analysis and the obtained results were classified into V parts. In the II section, LWF was introduced; in the III part, the way of projecting P by using LWF was explained, and the criteria for the selection of the amplification coefficient of P and dominant poles were provided, too. In the subsection A, FOTDS results were shown, while the subsection B displayed IFORDS results. The response of the control systems functioning under parameter P was presented in section IV; in the V part, the conclusion with a suggestion for further analysis was outlined.

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II. Lambert W Function

Lambert W function $W(z)$ is an equation solution

$$W(z)e^{W(z)} = z, z \in C, \qquad (1)$$

where C is a set of complex numbers. $W(z)$ has an infinite number of solutions and an infinite number of branches $W_k(z)$ where $k \epsilon (-\infty, \infty)$. If $z \epsilon R$, where R is a set of real numbers, only two branches $W_k(z)$ have real solutions (1) and, only if $z \geq 0$, then there is one solution $W_0(z)$; for $-1/e \leq z \leq 0$, there are two solutions $W_0(z)$ and $W_{-1}(z)$, Fig. 1.

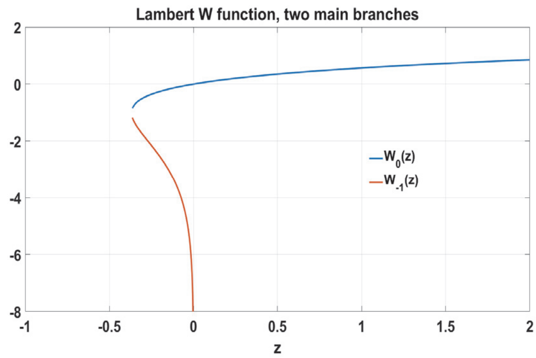

Fig. 1. A graph of function $W_k(z)=z$ for $z \epsilon R$, $k \epsilon (-1,0)$

The range of branches $W_k(z)$ and the conditions for the convergence into C, and various numerical methods for solving problems (1), were explained in [9]. For solving a scalar and matrix shape (1), Maple and Mathlab can be used.

III. Determining Parameter P of the Controller

Let the transfer functions be: a FOTDS object $G_p(s)$ (2), an IFOTDS object $G_{ip}(s)$ (3), P regulator $G_c(s)$ (4), where K_m are amplification coefficient of the FOTDS and IFOTDS, T_m their time constants, h time delay and K_p the amplification of P controller.

$$G_p(s) = \frac{K_m}{T_m s + 1} e^{-hs}. \quad (2)$$

$$G_{ip}(s) = \frac{K_m}{s(T_m s + 1)} e^{-hs}. \quad (3)$$

$$G_c(s) = \frac{U(s)}{E(s)} = K_p. \quad (4)$$

$y(t)$, $u(t)$, $e(t)$ and $r(t)$ denote the output of the coupled system with the negative feedback of the unit, which has $G_c(s)G_p(s)$ or $G_c(s)G_{ip}(s)$ in the direct branch, control signal as a passage into the object $G_p(s)$ or $G_{ip}(s)$ and concurrently as an output from the P controller $G_c(s)$, input signal into $G_c(s)$, a reference signal, and their Laplace transforms $Y(s)$, $U(s)$, $E(s)$ and $R(s)$, where:

$$E(s) = R(s) - Y(s), \quad (5)$$

then the closed-loop transfer with the regulator:

$$T_{p/ip}(s) = \frac{Y(s)}{R(s)} = \frac{G_c(s)G_{p/ip}(s)}{1 + G_c(s)G_{p/ip}(s)}. \quad (6)$$

A. Determining P parameter for the FOTDS model

By replacing (2) and (4) in (6) the equation below is obtained:

$$T_p(s) = \frac{Y(s)}{R(s)} = \frac{K_m K_p e^{-hs}}{T_m s + 1 + K_m K_p e^{-hs}}. \quad (7)$$

A characteristic system equation (8)

$$T_m s + 1 + K_m K_p e^{-hs} = 0, \quad (8)$$

has an infinite number of solutions. Translating it into a domain of time, a first-order DDE (9) for $r(t)=0$ is obtained.

$$\frac{dy(t)}{dt} + \frac{1}{T_m} y(t) + \frac{K_m K_p}{T_m} y(t-h) = 0. \quad (9)$$

Using simple mathematical operations, (8) is translated into a LWF shape (10), wherefrom (11) is obtained for the purpose of determining poles of the systems.

$$h(s + \frac{1}{T_m}) e^{h(s+\frac{1}{T_m})} = -\frac{K_m K_p h}{T_m} e^{\frac{h}{T_m}}. \quad (10)$$

$$s_k = \frac{1}{h} W_k\left(-\frac{K_m K_p h}{T_m} e^{\frac{h}{T_m}}\right) - \frac{1}{T_m}. \quad (11)$$

The selection of the desired poles, in the infinite spectrum of poles, is reduced to the selection of the dominant poles (the closest to the imaginary axis from the complex plane), and it cannot be arbitrary, but the target poles need to meet certain criteria. If λ_k is a solution of (11), then $\lambda_k = R_e\{\lambda_k\} + iI_m\{\lambda_k\}$. In order for the system to be stable, a condition $R_e\{\lambda_k\}<0$ must be satisfied; and it was established in [8] and [9] that the dominant poles are obtained by solving (11) only if $k\epsilon(-1,0)$. If it is (12) then [9] is (13)

$$W_k\left(-\frac{K_m K_p h}{T_m} e^{\frac{h}{T_m}}\right) = W_k(z) = \xi + i\eta, z = x + iy, \quad (12)$$

$$x = e^{\xi}(\xi \cos\eta - \eta \sin\eta),\ y = e^{\xi}(\eta \cos\eta + \xi \sin\eta). \quad (13)$$

Considering that $z = -\frac{K_m K_p h}{T_m} e^{\frac{h}{T_m}} \in R$, it follows that $y = e^{\xi}(\eta\cos\eta + \xi\sin\eta) = 0$, where: $\eta = 0 \vee \xi = -\eta\cot\eta$.

It is, from LWF characteristics, inferred that the solution (11), and thereby the stability of the system, taking into consideration that $h>0$, $K_m>0$, $T_m>0$, is dependent on a number of factors:

for z>0, it can be said that:

a) $z>(h/T_m)exp(h/T_m), W_0(z)>h/T_m$, $R_e\{\lambda_0\}>0$ system is unstable
b) $0<z<(h/T_m)exp(h/T_m)$, $0<W_0(z)<h/T_m$, $R_e\{\lambda_0\}<0$, $I_m\{\lambda_0\}=0$, $K_p<0$

for z<0, it follows that $K_p>0$ and:

c) $-1/e<z<0$, $-1<W_0(z)<0$, $-1<\xi<0$, $\eta=0$, $-1/h-1/T_m<R_e\{\lambda_0\}<-1/T_m$, $I_m\{\lambda_0\}=0$
d) $-\pi/2<z<-1/e$, $-1+i0<W_0(z)<0+i1.5708$, $\xi<0$, $\eta\neq0$ $-1/h-1/T_m<R_e\{\lambda_0\}<-1/T_m$, $0<I_m\{\lambda_0\}<i1.5708$
e) $-\pi<z<-\pi/2$, $0+i1.5708<W_0(z)<0.5008+i1.8369$, $\xi>0$, $\eta\neq0$, $-1/T_m<R_e\{\lambda_0\}<-1/T_m+0.5008/h$, $I_m\{\lambda_0\}\neq0$

from the criterion e) what clearly ensues is that $R_e\{\lambda_0\}<0$, if and only if $1/T_m>0.5008/h$ for each $W_0(z)$, or if $\xi<h/T_m$ which further implies that $W_0(z)<h/T_m+i\eta$, where η must fulfill $-\eta cot\eta=\xi$, which is also a sufficient condition for the stability in this range. From (13) $z=x$ is obtained, wherefrom critical K_p is received, which has the system oscillating in a constant amplitude with its own oscillation frequency. From the above it can be said that: if $-1/e<z<0$, the system is asymptotically stable and closed and where criteria (12) are valid.

$$0 < K_p < \frac{T_m}{ehK_m} e^{\frac{-h}{T_m}}, -\frac{1}{h} - \frac{1}{T_m} < \lambda_o < -\frac{1}{T_m}, \quad (12)$$

for $-\pi/2<z<-1/e$, the conditions are (13)

$$\frac{T_m}{ehK_m} e^{\frac{-h}{T_m}} < K_p < \frac{\pi T_m}{2hK_m} e^{\frac{-h}{T_m}}, \quad -\frac{1}{h} - \frac{1}{T_m} < \lambda_o < -\frac{1}{T_m} + \frac{i1.5708}{h}, \quad (13)$$

in which case the conditions of the range stability $-\pi<z<-\pi/2$ are already given.

During the P controller projection, the sufficient conditions for the selection of K_p and λ are (12), (13) and the conditions are shown in e).

B. Determining Parameter P for the IFOTDS Model

By replacing (3) and (4) in (6) the equation below is received

$$T_{ip}(s)=\frac{Y(s)}{R(s)}=\frac{K_m K_p e^{-hs}}{T_m s^2+s+K_m K_p e^{-hs}}, \tag{14}$$

Wherefrom a second-order DDE (15) for $r(t)=0$ is obtained by translating it into a time domain.

$$\frac{d^2y(t)}{dt^2}+\frac{1}{T_m}\frac{dy(t)}{dt}+\frac{K_m K_p}{T_m}y(t-h)=0. \tag{15}$$

By replacing $y=x_1, x_2=x_1'$, in (15)

$\frac{dx}{dt}=Ax(t)+A_d x(t-h)$, is obtained, where X, A, A_d (16)

$$x=(x_1,x_2),^T\ A=\begin{pmatrix}0 & 1\\ 0 & -\frac{1}{T_m}\end{pmatrix}, A_d=\begin{pmatrix}0 & 0\\ -\frac{K_m K_p}{T_m} & 0\end{pmatrix}. \tag{16}$$

A characteristic system equation (17), is reduced to a Lambert W shape (18). The introduced unknown matrix $Q\epsilon C^{2x2}$, has to satisfy (19). The solution (17) is received (20) if the condition (19) is met.

$$S-A-A_d e^{-hS}=0. \tag{17}$$

$$W_k(A_d hQ_k)e^{W_k(A_d hQ_k)}=h(S_k-A). \tag{18}$$

$$W_k(A_d hQ_k)e^{(W_k(A_d hQ_k)+Ah)}=A_d h. \tag{19}$$

$$S_k=\frac{1}{h}W_k(A_d hQ_k)+A. \tag{20}$$

The selection of the target poles S_k, in the infinite pole spectrum, cannot be arbitrary, but it presents the selection of the dominant poles (closest to the imaginary axis from the complex plane); and in [8] it was demonstrated that they are obtained by solving (20) only for $k=0$ or $k=\pm 1$, whereby dominance is guaranteed. By solving (19) and (20) for the desired poles, parameter P of the controller is determined.

IV. Characteristics of the Received Systems

C. Characteristics of the FOTDS model

Object $G_p(s)$ (2), which is being elaborated on, described in the part 3.2 [12] has parameters K_m=1, T_m=30, h=1. Based on the wanted system dynamics, a closed-loop or an oscillating system, we select poles so that they fulfill the criteria (12), (13) or e), and then we calculate the parameter K_p from (11).

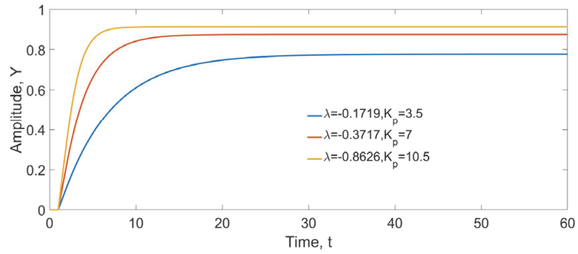

Fig. 2. The amplitude response for different λ values

By drawing the dominant pole near the imaginary axis, at closed-loop systems Fig. 2, which fulfill (12), the system reaction speed is reduced as well as K_p, and from the figure it can be clearly noticed that the system is not an oscillating system, which confirms the accuracy of (12).

For a desired oscillatory response Fig. 3, moving the dominant pole towards the imaginary axis, and fulfilling the condition (13), leads towards the increased maximum overshoot and the longer settling time, in which case K_p increases.

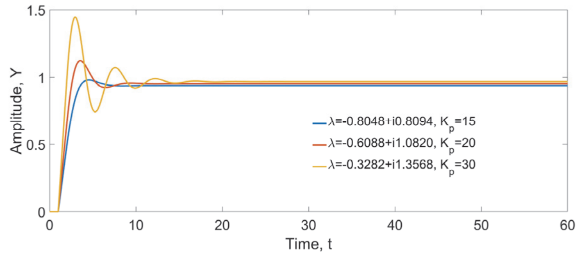

Fig. 3. The response of the oscillatory system

A borderline case λ=1.5917i results in K_p=47.7625. The received system is borderline stable, oscillating with the constant amplitude and natural oscillation frequency Fig. 4.

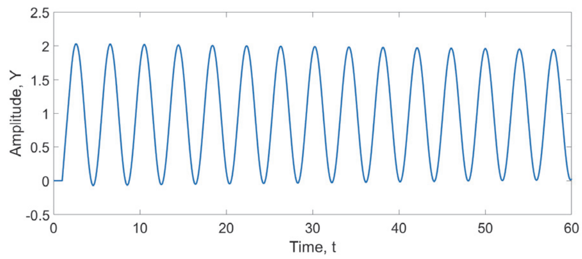

Fig. 4. The response of the system on the borderline of stability λ=1.5917i

The regular selection of K_p leads to a stable system of the desired dynamics. The fact that FOTDS is controlled in this way shows the error of tracing the reference signal in a stable state, which, for some systems, is not of the greatest significance, while for some other systems PI or PID controllers need to be used.

D. Characteristics of the IFOTDS model

Object $G_{ip}(s)$ (3), which is being dealt with, described in the section 2.2 [11] has parameters K_m=1, T_m=1, h=0.5. By replacing the values in (16) matrices A i A_d were obtained. Considering that the matrix A_d has a double zero value, it ensues that the matrix $A_d hQ$, for any value Q, will assume the following shape:

$$A_d hQ = \begin{pmatrix} 0 & 0 \\ m_{21} & m_{22} \end{pmatrix},$$

Therefore, a distinctive value $A_d hQ$ is equal to zero, which is why because of the characteristics of LWF [9], $W_k(0)=0$ for $k=0$, $W_k(0)=-\infty$ for $k\neq 0$, a traditional procedure of making use of identical branches $k_1=k_2$ cannot be applied, but rather hybrid branches $k_1\neq k_2$ are used in this case. By determining that $k_2=0$ and $k_1\epsilon(-\infty, \infty)$ the following is obtained:

$$W_{k_1k_2}(A_d hQ) = V \begin{pmatrix} W_{k_1}(m_{22}) & 0 \\ 0 & W_{k_2}(0) \end{pmatrix} V^{-1},$$

where V is a distinctive vector of the matrix $A_d hQ$. The desired dynamics of the system is established by selecting the dominant poles, whereby solving (19) and (20), K_p is received.

A system without a controller $K_p=1$, Fig. 7 displays the settling time of 15.5s, 45% of overshoot, and GM=6.64 dB PM=29.3 deg, Fig. 8.

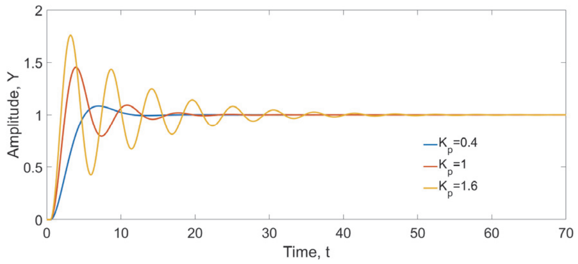

Fig. 7. An overshoot response of the system for various K_p values

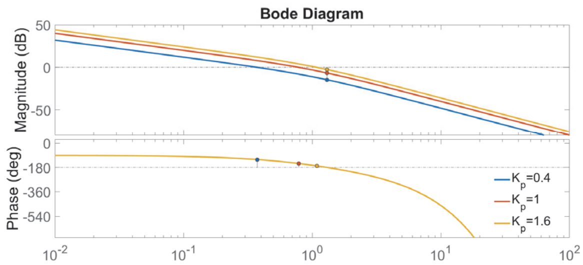

Fig. 8. A frequency response of the open-loop system

By moving a pole onto the left side of the complex plane, which reduces K_p, what is received are the overshoot and the settling time thus, if the projection for PM of around 60 degrees is performed, $K_p=0.4$, the settling time of 10.2 and the overshoot smaller than 9%, will be obtained Fig. 7.

By moving the pole towards the imaginary axis from the complex plane, the oscillation of the system is increased, an increased overshoot and the longer settling time are obtained. For $K_p=1.6$, Fig. 7, the overshoot is 76% and the settling time is 39 s.

It is quite clear that due to the first-order IFODTS astatism the system's steady state error equals zero, unlike in FOTDS.

V. Conclusion

P controller designed with LWF can be used for controlling the FOTDS and IFOTDS, if the systems are affected by minor perturbation, or if the steady state error of the system in the presence of perturbation does not greatly affect the functioning of the system. Otherwise, PI or PID should be used.

In the future, what can be explored is the effect of the perturbation on this kind of controlled system, as well as the projection of PI and PID controllers for the IFOTDS by using the Lambert W method, and the projection of PID controllers for the FOTDS.

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