# The $q$-Exponential Functions and Orthogonal Polynomials as the Special Motzkin Paths

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***Abstract*** – **The exponential function, since its introduction, was a subject of a lot of variations, deformations and generalizations. The same happened with different classes of orthogonal polynomials. Here, we wish to emphasize their appearance in some combinatorial problems. The Motzkin paths are a class of positive weighted paths with the appearance in many contexts. We will connect them with the continued fractions, the moments of the functionals and generalizations of the exponential function. Such paths have their interpretations in the computer sciences.**

*Keywords* – **Paths, Combinatorics, Exponential functions, Orthogonal Polynomials.**

## I. Introduction

The very beginnings of a combinatorial theory of special functions is given in Shapiro's paper [1] in 1981. and Viennot's paper [2] in 1983. Nowadays we can note growing interest in this area, a lot of books (G. Andrews [3] and Stanley [4]), papers and thesis were done with this topic (A.T. Benjamin [5] and D. Drake [6]). This method is used in [5] in elementary combinatorial proofs for special numbers identities and combinatorial interpretation of famous Rogers-Ramanujan identities in [7] by D.P. Little and J.A. Sellers. A lot of nice applications for multiple and quasi-orthogonal polynomials, then $q$-orthogonal polynomials can be found (D. Stanton [8]).

## II. On q-Exponential functions

In the theory of $q$-calculus (Andrews [3], Stanley [4], Stanton [8] or Rajković-Marinković-Stanković [9]), for a real parameter $q \in (-1, 1)$, we introduce a $q$-real number $[a]_q$ by

$$[a]_q = \frac{1-q^a}{1-q} \qquad (a \in \mathbb{R}). \tag{1}$$

The $q$-analog of the Pochhammer symbol ($q$-shifted factorial) is defined by:

$$(a;q)_0 = 1, \quad (a;q)_k = \prod_{j=1}^{k} (1 - aq^{j-1}),\ k \in \mathbb{N} \cup \{\infty\}, \tag{2}$$

and

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(a\, q^\alpha;q)_\infty} \quad (\alpha \in \mathbb{R}). \tag{3}$$

In the Euler's works, we can find two analogs of exponential function

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{(q;q)_n} = \frac{1}{(x;q)_\infty} \quad (|x| < 1), \tag{4}$$

$$E_q(x) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} x^n}{(q;q)_n} = (-x;q)_\infty, \quad (0 < |q| < 1). \tag{5}$$

Having in mind that

$$\lim_{q\to 1} (q;q)_k = \lim_{q \to 1} \prod_{j=1}^{k} (1-q^j) = (1-q)^k k!, \quad k \in \mathbb{N}, \tag{6}$$

it is valid

$$\lim_{q\to 1} e_q((1-q)x) = \lim_{q \to 1} E_q((1-q)x) = e^x. \tag{7}$$

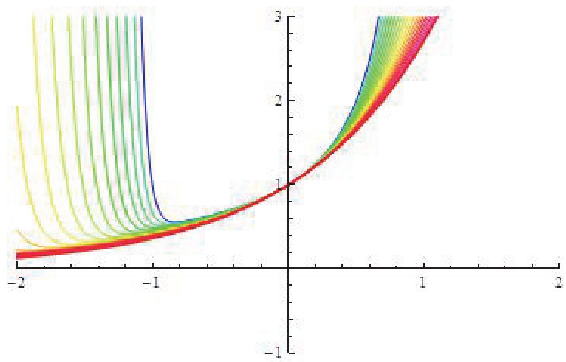

Fig. 1. The graphics of the small $q$-exponential function $e_q((1-q)x)$ for $q = 0(0.1)1$.

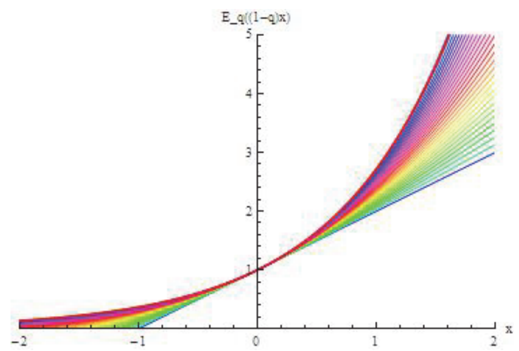


Fig. 2. The graphics of the big $q$-exponential function $E_q((1-q)x)$ for $q = 0(0.1)1$.

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These functions somehow retain some important properties of exponential function, such as, for example,

$$e_q(x)E_q(-x)=1. \tag{8}$$

We can express

$$E_q((1-q)x)=\sum_{n=0}^{\infty}E(n,q)x^n, \tag{9}$$

where

$$E(n,q)=\frac{q^{\binom{n}{2}}(1-q)^n}{(q;q)_n}=\frac{q^{\binom{n}{2}}}{\prod_{k=1}^{n-1}\sum_{i=0}^{k}q^i}. \tag{10}$$

Hence we have the estimation:

$$|E(n,q)|<q^{\binom{n}{2}}, \tag{11}$$

wherefrom, for 0< $q$, $|x|$ < 1, it is valid

$$|E_q((1-q)x)|<1+|x|+\frac{qx^2}{1-|x|\sqrt{q}}. \tag{12}$$

Exton [10] has defined a generalization of $q$-exponential function by

$$E^{(\text{Exton})}(q,\lambda;z)=\sum_{n=0}^{\infty}\frac{q^{\lambda n(n-1)}}{[n]_q!}z^n \quad (\lambda\in R; z\in C). \tag{13}$$

Floreanini and coauthors [11] have considered the one-parameter family of $q$-exponential functions of the type

$$E_q^{(\text{Fl})}(z;\alpha)=\sum_{n=0}^{\infty}\frac{q^{\alpha n^2/2}}{(q;q)_n}z^n \qquad (\alpha\in R; z\in C). \tag{14}$$

Here, it is true

$$\lim_{q\to 1}E_q^{(\text{Fl})}((1-q)z;\alpha)=e^z \qquad (z\in C). \tag{15}$$

These functions are connected by

$$E^{(\text{Exton})}(q,\lambda;z)=E_q^{(\text{Fl})}\left(\frac{1-q}{q^{\lambda}}z;2\lambda\right) \qquad (\lambda\in R; z\in C). \tag{16}$$

Again, we notice two well-known particular cases of this family:

$$e_q(z)=E_q^{(\text{Fl})}(z;0),\quad E_q(z)=E_q^{(\text{Fl})}\left(\frac{z}{\sqrt{q}};1\right). \tag{17}$$

Today, there are a few new deformations and generalizations of the exponential function very useful in various areas of science, such as mechanical statistics, for example in Stanković-Marinković-Rajković [12].

## III. Basics about Continued Fractions

For two real sequences $R=\{r_n\}$ and $S=\{s_n\}$ we can define the *continued fraction* $K=K(S,R)$ like a sequence $\{K_n\}$ with an arbitrary member (Flajolet [13]):

$$K_n=s_0+\cfrac{r_1}{s_1+\cfrac{r_2}{s_2+\cfrac{r_3}{s_3+\ddots+\cfrac{r_n}{s_n}}}}. \tag{18}$$

The term $K_n$ is the $n$th convergent of the continued fraction which value is

$$K=\lim_{n\to\infty}K_n. \tag{19}$$

The expression (1) requires a lot of spacing. That is why the $n$th convergent is often written in the form

$$K_n=s_0+\frac{r_1|}{|s_1}+\frac{r_2|}{|s_2}+\cdots+\frac{r_n|}{|s_n}. \tag{20}$$

We can simplify $K_n$ into the form

$$K_n=\frac{N_n}{D_n}, \tag{21}$$

where $N_n$ is $n$th partial numerator and $D_n$ is $n$th partial denominator determined by

$$N_n=s_nN_{n-1}+r_nN_{n-2},\quad D_n=s_nD_{n-1}+r_nD_{n-2} \tag{22}$$

with initial values $N_{-1}=1, N_0=s_0$, and $D_{-1}=0, D_0=1$.

Some authors write

$$K=s_0+\mathrm{K}_{i=1}^{\infty}\frac{r_i}{s_i}. \tag{23}$$

**Example 3.** The transformation of a series into its equivalent continued fraction, with the series partial sums being equal to the continued fraction convergents, is due to Euler. The series $G=\sum_{n=0}^{\infty}g_n$ is transformed into the continued fraction $K=s_0+\mathbf{K}_{i=1}^{\infty}\frac{r_i}{s_i}$, by the relations:

$$s_0=g_0,\quad r_1=g_1,\quad s_1=1, \tag{24}$$

$$r_n=-\frac{g_n}{g_{n-1}},\quad s_n=1+\frac{g_n}{g_{n-1}},\quad n\geq 2. \tag{25}$$

**Example 3.** The exponential function $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$, has $d_n=\frac{x^n}{n!}$, wherefrom $r_n=-\frac{x}{n}$, $s_n=1+\frac{x}{n}$, $n\geq 2$.

Hence $e^x$ can be represented by the continued fraction

$$e^x = 1 + \cfrac{x}{1 - \cfrac{x/2}{1 + x/2 - \cfrac{x/3}{1 + x/3 - \cfrac{x/4}{1 + x/4 - \cdots}}}}. \tag{26}$$

Applying an equivalence transformation that consists of clearing the fractions this example is simplified to

$$e^z = \cfrac{1}{1 - \cfrac{z}{1 + z - \cfrac{z}{2 + z - \cfrac{2z}{3 + z - \cfrac{3z}{4 + z - \ddots}}}}}.$$

## IV. ALPHABET

A *letter* is a variable which is not non-commutative with others. An *alphabet* X is a the maximal set of different letters. A *word* is a linear combination of the letters, i.e. a word is formed from the *concatenation* $U$ of finitely many letters. A set of all finite words we define by $X^*$. Hence

$$U(u_1, u_2, \cdots, u_m) = u_1 u_2 \cdots u_m \in X^* \qquad (u_1, u_2, \cdots, u_m \in X). \tag{27}$$

The number of words with length is $n$ is $(\text{card}(X))^n$. The concatenation $U$ is very similar to the set union, but it is not the same operation.

**Example 1.** For the binary alphabet $B = \{0,1\}$, its norm is $\|\{0,1\}\| = 2$, and $\{0,1\}^*$ is the set of binary words

$$\{0,1\}^* = \{\varnothing, 0, 1, 00, 10, 01, 11, \ldots\}. \tag{28}$$

**Example 2.** The set of all words made via the alphabet $\{a,b,c\}$ is

$$\{a,b,c\}^* = \{\varnothing, a, b, c, aa, ab, ac, ba, bb, bc, \ldots\}. \tag{29}$$

**Example 3.** In a genomic sequence, the alphabet is $\{A, G, C, T\}$.

A *monoid* $(Y,+)$ is an algebraic structure with a single associative binary operation and an identity element.
A *monoid ring* $(Y,+,*)$ over $R$ is the set of formal sums

$$s = \sum_{u \in Y} s_u u \quad (s_u \in R; u \in Y) \tag{30}$$

with operations:

$$s + t = \sum_{u \in Y} (s_u + t_u) u, \tag{31}$$

$$s * t = \sum_{u \in Y} \left( \sum_{v * w = u} s_v t_w \right) u \quad (s_u, t_u, s_v, t_w \in R). \tag{32}$$

Let $u = u_1 u_2 \cdots u_m \in X^*$ and $v = v_1 v_2 \cdots v_n \in X^*$ be two words. The sum of two words is their union and the products are their concatenation:

$$u + v = u \cup v \in X^*, \quad u * v = u_1 u_2 \cdots u_m v_1 v_2 \cdots v_n \in X^*.$$

Hence $(X^*, +, *)$ over the set of real numbers $R$ is a monoid ring.

Formal power series extend the usual algebraic operations on polynomials to infinite series of the form

$$f(z) = \sum_{n=0}^{\infty} f_n z^n. \tag{33}$$

The set of all power series with operations $(C[z], +, \cdot)$ forms the ring.

It is possible to define a morphism $\mu : X^* \to C[z]$. We will deal with it in the next sections.

To the infinite alphabet

$$X = \{a_j\}_{j \in N_0} \cup \{b_j\}_{j \in N_0} \cup \{c_j\}_{j \in N_0}. \tag{34}$$

Flajolet has joined the next $n$th convergent:

$$F_n = \cfrac{1}{1 - b_0 - \cfrac{a_0 c_1}{1 - b_1 - \cfrac{a_1 c_2}{1 - b_2 - \ddots_{1 - b_{n-1} - \cfrac{a_{n-1} c_n}{1 - b_n}}}}}. \tag{35}$$

## V. MOTZKIN PATHS

Let $M_k = (x_k, y_k)$ $(k=0,1,\ldots n)$ be the points in $Oxy$ plane with $y_k$ nonnegative. Then $P_0 P_1 \ldots P_n$ is a *Motzkin path* if $P_0 = (0,0)$, $P_n = (x_n, 0)$, and vector $u_{k+1} = \overrightarrow{P_k P_{k+1}}$ is one of the next three vectors (up, horizontal and down vector):

$$a = \overrightarrow{OA}, \quad b = \overrightarrow{OB}, \quad c = \overrightarrow{OC}, \tag{36}$$

where

$$O(0,0), \quad A(1,1), \quad B(1,0), \quad C(1,-1). \tag{37}$$

So, a path can be considered like sequence $u = u_1 u_2 \cdots u_n$. The *length* of the path is number $n$ and its *height* is $h(u)=\max\{y_k\}$.
The *level steps* of path are the vectors $b$. The *area below a path* is the sum of the height of the points in the path. The weight $w(k)$ of a vector $u_k$ is the number which depends on its type:

$$w(k) = \begin{cases} a_{y_{k-1}}, & \text{if } u_k = a, \\ b_{y_{k-1}}, & \text{if } u_k = b, \\ c_{y_{k-1}}, & \text{if } u_k = c. \end{cases} \tag{38}$$

In common words, it is $a$, $b$ or $c$ with the sub-index which is equal to the ordinate of its starting point.

Denote by $\mathbf{M}_n$ the set of all Motzkin paths of length $n$. The number $M_n = |\mathbf{M}_n| = \mathrm{card}(\mathbf{M}_n)$ is the $n$-th Motzkin number.

**Example 3.** All Motzkin paths of size $n = 3$. That is why $M_3 = 4$.

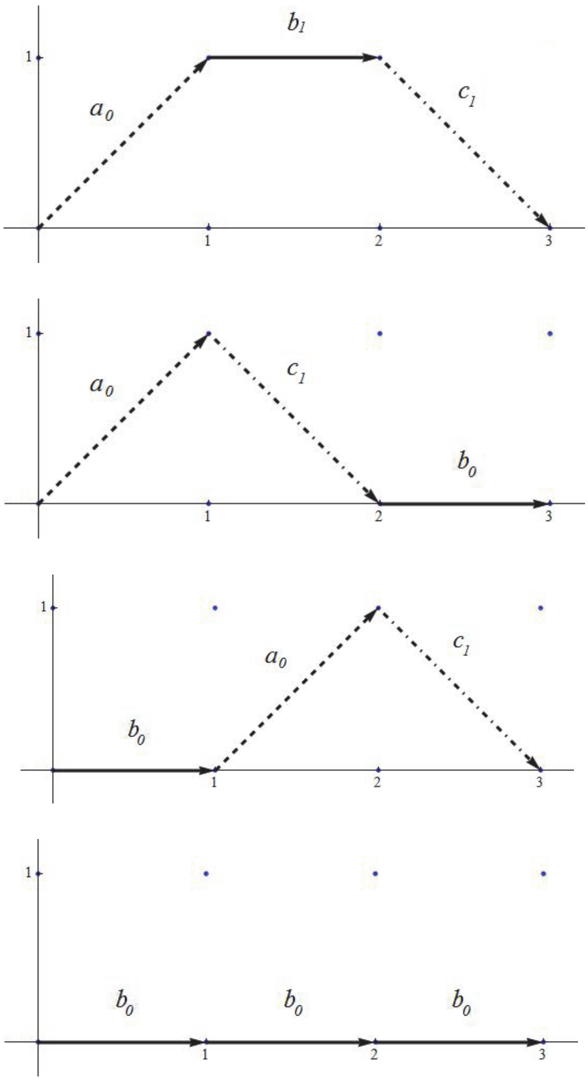


Fig. 3. All Motzkin paths of size $n = 3$.

The Motzkin paths are of the form

$$M = \varnothing + aM + bMcM. \tag{39}$$

Translating the equation above to generating functions, we get

$$M(x) = 1 + xM(x) + x^2M^2(x), \tag{40}$$

i.e.,

$$M(x) = \frac{1 - x - \sqrt{(x-1)^2 - 4x^2}}{2x^2}. \tag{41}$$

The Motzkin numbers are $\{1, 1, 2, 4, 9, 21, 51, 127, 323, \ldots\}$.

The *weight w(P) of a path P* is the product of the weights of its segments, i.e.

$$w(P) = \prod_{k=1}^{n} w(k). \tag{42}$$

**Example 2.** The path on the Fig. 4, has the length $n = 8$, the height $h = 2$, and the weight $w = a_0b_1a_1c_2c_1b_0a_0c_1$, and area $= 1 + 1 + 2 + 1 + 1 = 6$.

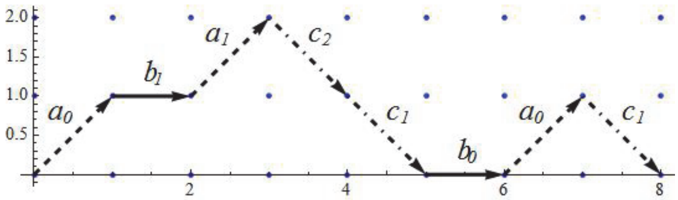


**Fig. 4.** A Motzkin path of size $n = 8$.

By convention, the empty path will have weight 1.
Let $\theta_n$ be the set of all weighted paths with height at most $n$.
The *total weight* of $\theta_n$ is denoted by

$$G_n = \sum_{T \in \theta_n} w(T). \tag{43}$$

Some authors use term the *generating function* for (7) which depends on $n$ and tiles weights. Especially, when weights depends of the same variable, we will get the well-known notion of generating function in mathematics.

***Theorem 1.*** (Flajolet [13]) *The total weights of all weighted paths of the height n is given by Flajolet's nth convergent (6).*

A *Dyck path* of length $n$ is a Motzkin path without East steps, i.e. $b_k = 0 \quad (k \geq 1)$.

**Example 3.** A morphism $\mu : X^* \to C[z]$ defined by
$\mu(a_j) = zq^j \quad (j \geq 0), \quad \mu(b_k) = 0, \quad \mu(c_k) = zq^k \quad (k \geq 1)$
has the generating function

$$G(z,q) = \cfrac{1}{1 - \cfrac{z^2q^1}{1 - \cfrac{z^2q^3}{1 - \cfrac{z^2q^5}{1 - \cdots}}}}. \tag{44}$$

Hence

$$G(z,q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} z^n = E_q^{(\mathrm{FI})}(z;2). \tag{45}$$

**Example 4.** Let us define the weights

$$\mu(a_j)=z \quad (j\ge 0),\quad \mu(b_k)=0,\quad \mu(c_k)=q^k \quad (k\ge 1) \qquad (46)$$

Then we have the generating function

$$G(z,q)=\cfrac{1}{1-\cfrac{zq^1}{1-\cfrac{zq^2}{1-\cfrac{zq^3}{1-\cdots}}}}=\cfrac{1}{1-zq\cfrac{1}{1-\cfrac{(zq)q}{1-\cfrac{(zq)q^2}{1-\cdots}}}}, \qquad (47)$$

which satisfies the relation

$$G(z,q)=\frac{1}{1-qzG(zq,q)}. \qquad (48)$$

Suppose that $G(z,q)$ can be written in the form

$$G(z,q)=\frac{A(z,q)}{B(z,q)}=\frac{\sum_{n=0}^{\infty}a_n(q)z^n}{\sum_{n=0}^{\infty}b_n(q)z^n}.$$

Then, the relation (45) gets the form

$$\frac{A(z,q)}{B(z,q)}=\cfrac{1}{1-qz\cfrac{A(qz,q)}{B(qz,q)}}. \qquad (49)$$

Identifying numerators and denominators, we get

$$A(z,q)=B(qz,q),\quad B(z,q)=B(qz,q)-qzB(q^2z,q). \qquad (50)$$

From the second relation, we find

$b_0(q)=1,\quad b_n(q)=q^n b_n(q)-q^{2n-1}b_{n-1}(q),$

i.e.,

$$b_n(z,q)=\frac{(-1)^n q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}. \qquad (51)$$

Finally,

$$G(z,q)=\frac{E_q^{(\mathrm{FI})}(-qz;2)}{E_q^{(\mathrm{FI})}(-z;2)}. \qquad (52)$$

**Example 5.** The Rogers–Ramanujan continued fraction has the value:

$$RR(z,q)=1+\cfrac{zq}{1+\cfrac{zq^2}{1+\cfrac{zq^3}{1+\cfrac{zq^4}{1+\cdots}}}}=\frac{E_q^{(\mathrm{FI})}(z;2)}{E_q^{(\mathrm{FI})}(qz;2)}. \qquad (53)$$

**Example 6.** Let $S_n$ be the set of all permutations $\pi=\{p_1,p_2,\ldots,p_n\}$ over $n$ elements $\{1,\ldots,n\}$. A number $p_k$ id displaced if it not equal to $k$. For any permutation $\pi=\{p_1,p_2,\ldots,p_n\}$ its *distance* to the identity permutation as the sum of the displacements of all elements (Bartschi [14]):

$$D(\pi)=\sum_{k=1}^{n}\left|k-p_k\right|. \qquad (54)$$

The number of permutations of total displacement equal to $2d$ is

$$D(n,d)=\mathrm{card}\{\pi\in S_n:\quad D(\pi)\}=2d. \qquad (55)$$

These permutations are in correspondence to Motzkin paths whose area is exactly the distance $d$ under consideration. Therefore the problem translates into the problem of counting weighted Motzkin paths of a given area $d$.

## VI. Orthogonal Polynomials

The orthogonal polynomials are very useful tool in technical sciences. They arose in the 18th century in the study of mechanics, and after that they found applications in others. They appear in numerous mathematical disciplines such as: numerical analysis, approximation theory, differential equa-tions, combinatorics, number theory and statistics .

Let $\mu(x)$ be a positive Borel measure on an interval *(a,b)* with infinite support and such that all moments

$$\mu_n=\int_a^b x^n\, d\lambda(x)=L[x^n], \qquad (56)$$

exist. We take $\lambda(x)$ such that $\mu_0=1$. In this manner, we define linear functional $L$ in the linear space of real polynomials ***P***.

We can introduce an inner product as follows:

$$(f,g)=L[f\,g] \quad (f,g\in P)\,. \qquad (57)$$

which is positive-definite because of the property

$$\|f\|^2=(f,f)\ge 0 \quad (\forall f\in P). \qquad (58)$$

The polynomials $\{P_n(x)\}$ defined by

$$P_n(x)=\frac{1}{\det[\mu_{i+j}]_{i,j=0}^{n-1}}\begin{vmatrix}\mu_0 & \mu_1 & \cdots & \mu_n\\ \mu_1 & \mu_2 & & \mu_{n+1}\\ \vdots & & & \\ \mu_1 & & & \mu_{2n-2}\\ 1 & x & & x^n\end{vmatrix}, \qquad (59)$$

are *orthogonal* with respect to this inner product, i.e. the following holds:

$$(P_m,P_n)=0 \quad (m\ne n),\qquad \|P_n\|>0 \quad (\forall n). \qquad (60)$$

They satisfy the three-term recurrence relation

$$P_{k+1} = (a_k x - b_k)P_k - c_k P_{k-1}, \qquad P_{-1} = 0, P_0 = 1. \tag{61}$$

Conversely, if $\{a_j\}_{j\in N_0}$, $\{b_j\}_{j\in N_0}$ and $\{c_j\}_{j\in N_0}$ are known, we need an algorithm to find the moments. According to definition of the Motzkin paths, the moments are (Flajolet [13])

$$\mu_n = \sum_{P_n \in MP_n} w(P_n), \tag{62}$$

where $MP_n$ is the set of al Motzkin paths of length $n$. Especially, when $a_k = 1\ (\forall k)$, they are monic polynomials.

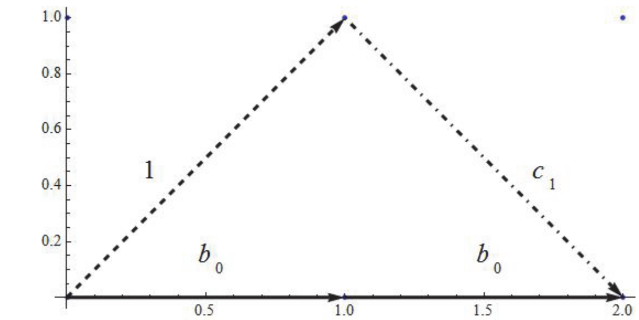


Fig. 5. The second moment $\mu_2 = b_0^2 + c_1$.

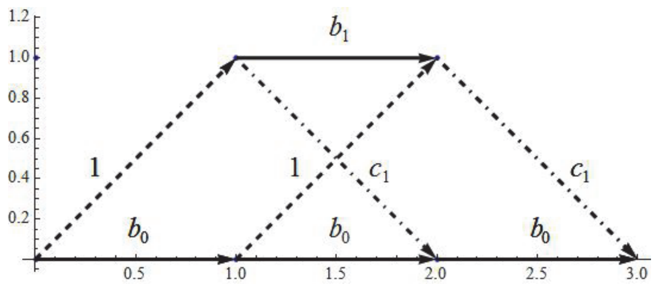


Fig. 6. The third moment $\mu_3 = b_0^3 + 2b_0c_1 + b_1c_1$.

Even more, we can determine their generating function

$$M(z) = \sum_{n=0}^{\infty} \mu_n z^n = \cfrac{1}{1 - b_0 z - \cfrac{c_1 z^2}{1 - b_1 z - \ddots - \cfrac{c_n z^2}{1 - b_{n-1} z - \cfrac{}{1 - b_n z - \cdots}}}}. \tag{63}$$

**Example 6.** The Motzkin paths with the weights

$$\mu(a_j) = 1, \quad (j \ge 0), \quad \mu(b_k) = 0, \quad \mu(c_k) = k \quad (k \ge 1) \tag{64}$$

have the moments

$$\mu_n = \sum_{P_n \in MP_n} w(P_n) = \begin{cases} (n-1)!!, & n-\text{even}, \\ 0, & n-\text{odd}. \end{cases} \tag{65}$$

Using (50), we get the Hermite polynomials:

$$H_{k+1} = xH_k - \frac{k}{2} H_{k-1}. \tag{66}$$

Very suggestive interpretations of this polynomials can be found in Drake's paper [6].

## VII. Conclusion

In this paper we have discussed the appearance of the generalizations of exponential functions and orthogonal polynomials in the combinatorics and their combinatorial interpretations. It seems that a lot of others can be expressed in that way.

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