Portfolio Risk Optimization Based on MVO Model

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Abstract – **This paper presents a portfolio risk optimization based on Markowitz's mean variance optimization (MVO) model. Historical return data for three asset classes are used to calculate the optimal proportions of assets, included in the portfolio, so that the expected return of each asset is no less than in advance given target value. Ten optimization problems are solved for different expected rate of return. The optimization is performed by a MATLAB solver.**

Keywords – **Portfolio optimization, mean variance optimization model, MATLAB.**

I. Introduction

Most investors and financial economists acknowledge the investment benefits of efficient portfolio diversification. Markowitz gave the classic definition of portfolio optimality: a portfolio is efficient if it has the highest expected (mean or estimated) return for a given level of risk (variance) or, equivalently, least risk for a given level of expected return of all portfolios from a given universe of securities. The portfolio optimization is a hard optimization problem in the finance area. There are developed different single objective optimization models for different applications in this area. This problem is very important. It is connected with the choice of a collection of assets to be held by an institution or a private individual. The choice should be done in such a way, that the expected return (mean profit) is maximized, while the risk is to be minimized at the same time. Dependent on users preferences, various trade-offs are usually seek.

The relatively low level of analytical sophistication in the culture of institutional equity management is one often-cited reason for the lack of acceptance of MV optimization, along with organizational and political issues.

Formulating this problem in optimization terms, Markowitz [1] states that, ideally, the investor searches for the **optimal portfolio**, i.e., the portfolio that minimizes the risk (within a defined tolerance) while maximizing the return.

II. Problem Formulation

The assets S_1 , S_2, … S_n ($n \geq 2$) with random returns are considered. Let a set of $n \in$ N financial assets be given. At time $t_0 \in$ R, each asset i has certain characteristics, describing its future payoff: Each asset i has an expected rate of return μ_i per monetary unit (e. g. dollars), which is paid at time $t_1 \in$ R, $t_1 > t_0$. Let $\mu = [\mu_1, \mu_2, \dots, \mu_n]^T$. This means if we take a position in $y \in$ R units of asset 1 at time t_0 our expected payoff in t_1 will be $\mu_1\, y$ units. Let σ_i be the standard deviation of the return of asset Si. For $i \neq j$, ρ_{ij} denotes the correlation coefficient of the returns of asset S_i and S_j. The correlation coefficient $\rho_{ii} = 1$. Let $\zeta = (\sigma_{ij})$ be $n\times n$ symmetric covariance matrix with $\sigma_{ii} = \sigma_i^2$ and $\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$ for $i \neq j$, and $i, j \in \{1, ..., n\}$. In this notation σ_{ii} is the variance of asset i-th's rate of return and σ_{ij} is the covariance between asset i-th's rate of return and asset j-th's rate of return.

A portfolio is defined by a vector $x := (x_1, ..., x_n) \in R^n$, which contains the proportions $x_i \in R$ of the total funds invested in securities $i \in \{1, ..., n\}$. The expected return and the variance of the resulting portfolio $x := (x_1, ..., x_n)$ can be presented (see standard Markowitz setting [2 - 7]) as follows:

$$f_1(x) = f_{\text{risk}}(x) = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \sigma_{ij} \qquad (1)$$

$$f_2(x) = f_{\text{return}}(x) = \sum_{i=1}^{n} x_i \mu_i \qquad (2)$$

It should be noted that $x^T\zeta x \geq 0$ for any x, since the variance is always nonnegative, i. e. ζ is positive semidefinite. It is assumed here, that ζ is positive definite, which is equivalent to assuming that there are no redundant assets in our collection $S_1, S_2, \dots, S_n$. Further it is assumed, that the set of admissible portfolios is a nonempty polyhedral set, represented as $X := \{x\!: Ax = b,\ Cx \geq d\}$, where A is an $m\times n$ matrix, b is m-dimensional vector, C is a $p\times n$ matrix and d is a p-dimensional vector. One constraint of type linear equality in the set X in the standard problem formulations is:

$$\sum_{i=1}^{n}(x_i) = 1 \qquad (3)$$

Simple linear inequality constraints (lower and upper bounds) in the set X are connected with the requirement, that the proportions (weights) of the portfolio should be nonnegative:

$$0 \leq x_i \leq 1,\ i \in \{1, ..., n\} \qquad (4)$$

There are several different single objective model formulations of Markowitz'mean-variance optimization (MVO) problem (see [7 - 10]). One single objective MVO model is formulated as follows:

$$\min_x \ \frac{1}{2} x^T \zeta x \qquad (5)$$

$$\text{subject to:} \quad \mu^T x \geq T \qquad (6)$$

$$Ax = b \qquad (7)$$

$$Cx \geq d. \qquad (8)$$

This model corresponds to risk minimization. In the first constraint T is a target value, where the expected return is no

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less than T. In case the problem formulation includes varying T between $T_{\min}$ and $T_{\max}$, there will be obtained efficient portfolios.

Defining Efficiency

The notion of defining an optimal set of portfolio weights to optimize risk and return is the basis of Markowitz portfolio efficiency. The efficiency criterion states:

A portfolio P* is MV efficient if it has least risk for a given level of portfolio expected return.

The MV efficiency criterion is equivalent to maximizing expected portfolio return for a given level of portfolio risk.

A portfolio P* is MV efficient if it has the maximum expected return for a given level of portfolio risk.

Which formulation of portfolio efficiency is used is a matter of convenience.

Optimization Constraints

Linear constraints are generally included in institutional MV portfolio optimization. For example, optimizations typically assume that portfolio weights sum to 1(budget constraint eq. 3) and are nonnegative (no-short-selling constraint eq.4). The budget condition is a linear equality constraint on the optimization. The no-short-selling condition is a set of sign constraints or linear inequalities (one for each asset in the optimization) and reflects avoidance of unlimited liability investment often required in institutional contexts. In practice, optimizations often include many additional linear inequality and equality constraints, particularly for equity portfolios.

Computer Algorithms

Several algorithms are available for calculating MV efficient portfolios. Quadratic programming is the technical term for the numerical analysis procedure used to compute MV efficient portfolio in practice. Quadratic programming algorithms allow maximization of expected return and minimization of the variance, subject to linear equality and inequality constraints (see [11 - 13]).

Many algorithms are used for computing MV efficient portfolios. The choice may depend on convenience, computational speed, number of assets, number and character of constraints, and required accuracy.

III. Assets Data Used

Equity portfolio optimization is typical application of MV optimization in asset management. The assets generally include broad asset categories, such as U.S. equities and corporate and government bonds, international equities and bonds, real estate, hedge funds, and venture capital. Sample means, variances, and correlations, based on monthly, quarterly, or annual historic data, may serve as starting points for optimization input estimates. In this study the Markovitz's MVO model is applied to the problem of constructing a long-only portfolio of *French stocks*, *US bonds* and *cash deposit*. Historical return data for these three asset classes are used to calculate the optimal proportions of assets, included in a portfolio, so that the expected return of each asset is no less than in advance given target value. It should be noted that the most MVO models combine historical data with other indicators such as earnings estimates, analyst ratings, valuation and growth metrics, etc. This study differs from the above mentioned approaches. It is focused on the price based estimates for expositional simplicity.

IV. Illustrative Example

The 10-year Treasury bond index (CBOE Interest Rate 10 Year T No (^TNX) for the returns on *bonds* and the EURONEXT 100 (^N100) index for the returns on *stocks*, are used. It is assumed that the cash is invested in a money market account whose return is the 1% – deposit *interest rate*. For each asset historic data are used including the annual times series for the "Total Return" from November 1999 through February 2018, e.g. for 220 months period (see [14, 15]).

Let the "Total Return" for asset i = 1,2,3; and $t = 0,...,t_f$ [months], be denoted by I_{it}. Here t = 0 corresponds to November 1999, and $t = t_f$ corresponds to February 2018. For each asset i the raw data I_{it}, $t = 0,...,t_f$, can be converted into rates of return r_{it}, $t = 0,...,t_f$ [months], by means of the formula:

$$r_{it} = \frac{I_{i,t} - I_{i,t-1}}{I_{i,t-1}} \qquad (9)$$

Let the random rate of return of asset i be denoted by T_i. From the historical data we can compute the arithmetic mean rate of return for each asset:

$$\bar{r}_i = \frac{1}{t_f}\sum_{t=1}^{t_f} r_{it} \qquad (10)$$

obtaining for the concrete example the result in Table1:

Table 1. Arithmetic mean rate of return for each asset

	Bonds	Stocks	Interest rate
Arithmetic mean $\bar{r}_{it}$	0,02024%	0,13%	0,084545%

Since the rates of return are multiplicative over time, it is preferred the geometric mean to be used instead of arithmetic mean. The *geometric mean* is the constant monthly rate of return, that needs to be applied in months t = 0 through $t = t_f - 1$, in order to get the compounded Total Return I_{itf}, starting from I_{i0}. The formula for the geometric mean is:

$$\mu_i = \left(\prod_{t=1}^{t_f}(1+r_{it})\right)^{\frac{1}{t_f}} - 1 \qquad (11)$$

The results for the test example are presented in Table 2:

Table 2. Geometric mean rate of return for each asset

	Bonds	Stocks	Interest rate
Geometric mean μ_i	0,36%	0,0123%	0,085%

Further the covariance matrix is computed:

$$\mathbf{cov}(T_i,T_j) = \frac{1}{t_f}\sum_{t=1}^{t_f}(r_{it}-\bar{r}_i)(r_{jt}-\bar{r}_j) \qquad (12)$$

For the test example considered the covariance matrix is presented in Table 3:

Table 3. Covariance matrix for the test example

Covariance	Bonds	Stocks	Interest rate
Bonds	0,0076611701	-0,00011479	-0,000000115
Stocks	-0,00011479	0,0023643199	0,0000000086
Interest rate	-0,000000115	0,0000000086	0,0000000020

The volatility of the rate of return on each asset is computed:

$$\sigma_i = \sqrt{\mathrm{cov}(T_i,T_j)} \ . \qquad (13)$$

The result is presented in Table 4:

Table 4. Volatility of the rate of return for each asset

	Bonds	Stocks	Interest rate
Volatility	0,0875	0,0486	0,0000447

Then the correlation matrix $\rho_{ij} = \frac{\mathrm{cov}(T_i,T_j)}{\sigma_i\sigma_j}$ is also computed (see Table 5):

Table 5. Correlation matrix for the test example

Correlation	Bonds	Stocks	Interest rate
Bonds	1.	-0,026993	-0,029453
Stocks	-0,026993	1.	-0,0039587
Interest rate	-0,029453	-0,0039587	1.

Using the covariance matrix from Table 3 the Quadratic Programming (QP) formulation of the portfolio optimization is:

$$\min f = [0{,}0076611701 x_B^2 + 2.(-0{,}00011479)\, x_B.\, x_S + 2.(-0{,}000000115) x_B.\, x_I + 0{,}0023643199 x_S^2 + 2.0{,}0000000086 x_S.\, x_I + 0{,}0000000020\, x_I^2 \ , \qquad (14)$$

subject to:

$$x_B + x_S + x_I \geq T$$
$$x_B + x_S + x_I = 1$$
$$x_B\,,\, x_S\,,\, x_I \geq 0$$

This problem is solved 10 times, correspondingly for rate of return T = 6%, T = 6,5%, ..., T = 10,5% with increments of 0,5% by means of ***fmincon*** solver of MATLAB "Optimization Toolbox" [16], using the *Interior pint* algorithm.

V. Test Results

Starting by T = 6%, after 68 iterations the result presented on Fig. 1 obtained.

The optimization results for all ten optimization problems with different T-values by means of ***fmincon*** solver of MATLAB "Optimization Toolbox" are summarized in Table 6.

The curve of efficient frontier is presented on Fig. 2. Every optimal portfolio calculated is presented as a triangle lying on the efficient frontier in the standard deviation / expected return plane.

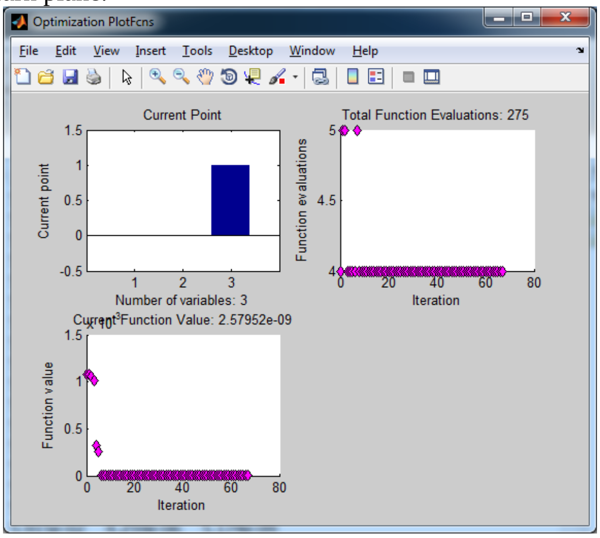

Fig. 1. Optimal portfolio for T = 6%

The optimization results for all ten optimization problems with different T-values by means of ***fmincon*** solver of MATLAB "Optimization Toolbox" are summarized in Table 1 as follows:

Table 6. Optimization results:

Objective function *f*	**Rate of return *T* [%]**	**Iterations**	**Total objective function evaluations**
2,57951566	6.	68	275
2,67812314	6.5	68	275
2,78430648	7.	68	291
2.89806094	7.5	66	270
3.01941047	8.	70	286
3.14832022	8.5	67	273
3.28481381	9.	67	279
3.42888482	9.5	66	271
3.58054119	10.	71	289
3.73975732	10.5	67	274

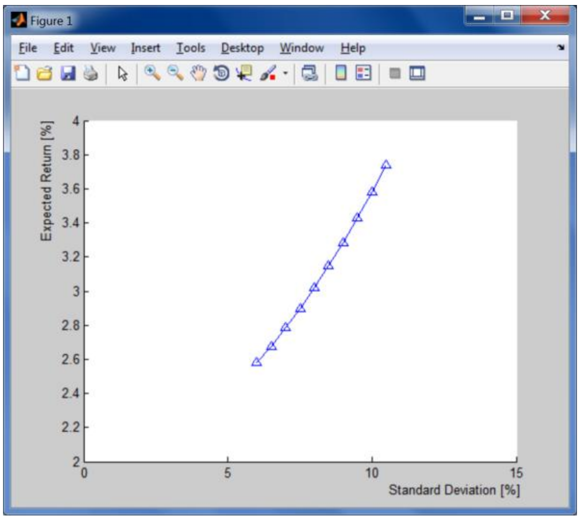

Fig. 2. The Efficient frontier curve

VI. Conclusion

Portfolio optimization of three classes of assets (bonds, stocks and cash deposit - interest rate) was performed. Historical return data (220 monthly returns) for these three asset classes are used to calculate the optimal proportions of assets, included in the portfolio, so that the expected return of each asset is no less than in advance given target value of return rate. Fig. 1. shows the results for an optimal portfolio by expected return of 6 %. Table 1 shows simulated optimizations results from 6 to 10,5 with step increase of 0.5% for the rate of return *T*. Through this simulation we get the efficient frontier for the alternative portfolios. The efficient frontier is often helpful in understanding the investments in the portfolio. It serves as useful guidepost for comparing the implications of different portfolios (see [17 - 19]). Fig. 2 shows the relationship between standard deviation and expected return. Logically, as the expected return increases, the deviation increases, too. Because of accepted pessimistic (conservative) strategy **1% positive increasing profitability** for a rate of return, the obtained result is that the bank deposit option is the most secure. We do not comment the security of investment in banks - how sure they are. Interesting investigation would be if a **negative interest rate on the bank deposit is set.** Additional experiments on a representative sample of benchmark problems should be performed for more reliable and precise conclusion about the efficiency and efficacy of the applied approach. Hopefully the mentioned challenges and potential directions for further research will attract more scientists to work in this fruitful area in the future.

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