

# Point charge in the air spherical hollow inside bi-isotropic material

Žaklina J. Mančić<sup>1</sup>, Zlata Ž. Cvetković<sup>2</sup> and Saša S. Ilić<sup>3</sup>

**Abstract** –In this paper, expressions for both electric and magnetic scalar potential of the point charge placed in air spherical hollow, inside bi-isotropic material of Tellegen type, are derived using the image theorem, which represents a contribution to the electromagnetic analysis of bi-isotropic materials.

**Keywords** –electrostatics, bi-isotropic material, image theorem, point charge.

## I. INTRODUCTION

Bi-isotropic materials have been incorporated in various electric devices and the research of specific problems is reported in a number of papers [1–5]. In previous years, authors have been reporting solutions of different problems, which incorporate these mediums, on several occasions, using EEM (Equivalent Electrode Method), which indirectly uses Image theorem for obtaining Green's function [6–8], or by applying numerical methods such as FEM (Finite Element Method) [9]. Furthermore, image theorem is exploited in the analysis of bi-isotropic body placed in homogeneous electric field [10] or for field calculation of charged ring near bi-isotropic sphere [11]. Bi-isotropic materials can be described using the following constitutive relations [1,2]

$$\mathbf{D} = \varepsilon \mathbf{E} + \xi \mathbf{H} \quad \text{and} \quad \mathbf{B} = \mu \mathbf{H} + \xi \mathbf{E} \quad (1)$$

where  $\mathbf{D}$  represents displacement field vector,  $\mathbf{E}$  is electric field vector,  $\mathbf{H}$  is magnetizing field vector,  $\mathbf{B}$  is magnetic field vector, while  $\varepsilon$  is permittivity of the material,  $\mu$  is permeability whereas  $\xi$  is the parameter which describes bi-isotropic characteristics of the material.

Such materials consist of elements that have permanent electrical and magnetic dipoles, parallel or antiparallel with others according to Tellegen. Thus, electric field in such material simultaneously orients both electrical and magnetic dipoles. Then as well, the magnetic field in such material orients both electric and magnetic dipoles at the same time. Laplace's equation for the electric scalar potential is obtained starting from Maxwell's equations and constituent relation (1):

$$\Delta \varphi = -\rho_s / \varepsilon_e, \quad \varepsilon_e = \varepsilon(1 - \xi^2 / \varepsilon \mu), \quad (2)$$

while Laplace's equation for the magnetic scalar potential

of free electric charges in the bi-isotropic environment is:

$$\Delta \varphi_m = \xi \rho_s / (\varepsilon_e \mu), \quad (3)$$

where  $\varepsilon \mu \neq \xi^2$  [12] and  $\Delta(\cdot)$  is the Laplacian function [13].

It is obligatory to integrate the Poisson's equations for the electric and magnetic scalar potential when determining the electric field in such environments, with respecting the existing boundary conditions. Then, field components are obtained using defining relations:

$$\mathbf{E} = -\text{grad} \varphi \quad \text{and} \quad \mathbf{H} = -\text{grad} \varphi_m. \quad (4)$$

In this paper, both theoretical field and potential calculation of the electric and magnetic scalar potential for the case of a point charge, placed into the air spherical hollow, is presented. It represents a contribution to the electrostatics of bi-isotropic materials.

## II. POINT CHARGE IN THE AIR SPHERICAL HOLLOW IN THE BI-ISOTROPIC MATERIAL- PROBLEM DEFINITION

If we set coordinate system such the charge is placed along  $z$ -axis while the center of air spherical hollow and the center of coordinate system coincide (Fig. 1), both electric and magnetic scalar potential inside the sphere satisfies Poisson's equation whereas the potentials outside the sphere satisfies Laplace's equation. The system is axially symmetrical and the potentials do not depend on  $\psi$ -coordinate of the spherical coordinate system [13].

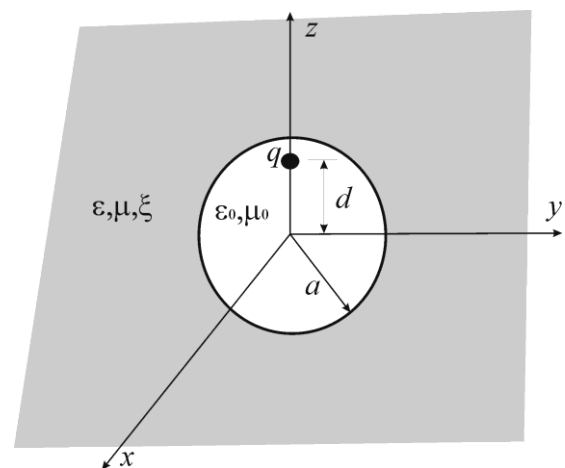


Fig. 1. Point charge in the air spherical hollow which is placed in bi-isotropic material.

<sup>1</sup>Žaklina J. Mančić, Faculty of Electronic Engineering, University of Niš, Aleksandra Medvedeva 14, 18106 Niš, Serbia, E-mail: zaklina.mancic@elfak.ni.ac.rs.

<sup>2</sup>Zlata Ž. Cvetković, Faculty of Electronic Engineering, University of Niš, Aleksandra Medvedeva 14, 18106 Niš, Serbia, E-mail: zlata.cvetkovic@elfak.ni.ac.rs

<sup>3</sup>Saša S. Ilić, Faculty of Electronic Engineering, University of Niš, Aleksandra Medvedeva 14, 18106 Niš, Serbia, E-mail: sasa.ilic@elfak.ni.ac.rs

The equation (2) for electric scalar potential is defined as:

$$\sin(\theta) \frac{\partial}{\partial r} \left( r^2 \frac{\partial \varphi}{\partial r} \right) + \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial \varphi}{\partial \theta} \right) = \begin{cases} -\frac{q\delta(r-d)\delta(\theta)}{2\pi\epsilon_0}, & r < a \\ 0, & r > a \end{cases} \quad (5)$$

whereas the equation for magnetic scalar potential is:

$$\sin(\theta) \frac{\partial}{\partial r} \left( r^2 \frac{\partial \varphi_m}{\partial r} \right) + \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial \varphi_m}{\partial \theta} \right) = 0, \quad 0 < r < \infty. \quad (6)$$

In (5)  $\delta$  is the Kronecker delta function. This is a function of two variables, usually just non-negative integers. The function is equal to one if the variables are equal ( $r = d$ ), and equal to zero otherwise.

The solution can be represented in the form:

$$\varphi = \sum_{n=0}^{\infty} R_n(r) P_n(\cos(\theta)), \quad (7)$$

where  $P_n(\cos(\theta))$  are Legendre polynomials of the first kind [13].

If we substitute equation (7) in equation (6), using Legendre differential equation:

$$\frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{dP_n}{d\theta} \right) = -n(n+1) \sin(\theta), \quad (8)$$

where  $P_n = P_n(\cos(\theta))$ , it is obtained:

$$\sum_{n=0}^{\infty} \left[ \frac{d}{dr} (r^2 R_n') - n(n+1) R_n \right] P_n(\cos(\theta)) \sin(\theta) = -\frac{q}{2\pi\epsilon_0} \delta(r-d) \delta(\theta) \quad (9)$$

If we multiply the previous equation with  $P_{n'}(\cos(\theta))$ , and integrate it with respect to  $\theta$ , from  $0$  to  $\pi$ , we obtain:

$$\frac{d}{dr} (r^2 R_n') - n(n+1) R_n = -\frac{q}{4\pi\epsilon_0} (2n+1) \delta(r-d), \quad (10)$$

taking into account that it is valid:

$$\int_0^{\pi} \sin(\theta) P_n(\cos\theta) P_{n'}(\cos\theta) d\theta = \begin{cases} 0, & n \neq n' \\ \frac{2}{2n+1}, & n = n' \end{cases} \quad (11)$$

and

$$P_n(1) = 1.$$

The equation (8) can be solved using the variation of constant method:

$$R_n(r) = \begin{cases} C_n r^n + D_n / r^{n+1}, & r \leq d \\ \left( C_n - \frac{q}{4\pi\epsilon_0 d^{n+1}} \right) r^n + \left( D_n + \frac{q d^n}{4\pi\epsilon_0} \right) \frac{1}{r^{n+1}}, & d \leq r \leq a. \\ A_n r^n + \frac{B_n}{r^{n+1}}, & a \leq r \end{cases} \quad (12)$$

Because the charge point is in the air, the equation (3) for magnetic scalar potential is:

$$\sin(\theta) \frac{\partial}{\partial r} \left( r^2 \frac{\partial \varphi_m}{\partial r} \right) + \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial \varphi_m}{\partial \theta} \right) = 0, \quad (13)$$

Analogously to the previous consideration, the solution for Eq (13) can be represented in the form:

$$\varphi_m = \sum_{n=0}^{\infty} R_{1n}(r) P_n(\cos(\theta)), \quad (14)$$

where function  $R_{1n}(r)$  is:

$$R_{1n}(r) = \begin{cases} C_{1n} r^n + D_{1n} / r^{n+1}, & r \leq a \\ A_{1n} r^n + \frac{B_{1n}}{r^{n+1}}, & a \leq r \end{cases}. \quad (15)$$

As the electric scalar potential  $\varphi$  in (12) has a finite value in the center of the sphere, as well as in the infinity, it has to be valid:

$$D_n = 0 \text{ and } A_n = 0. \quad (16)$$

Similarly, magnetic scalar potential  $\varphi_m$  has a finite value in the center of the sphere and in the infinity, thus:

$$A_{1n} = 0 \text{ and } D_{1n} = 0 \quad (17)$$

in (15).

The remaining constants can be determined from the boundary conditions for the continuity of the electric scalar potential:

$$\varphi(r = a - 0) = \varphi(r = a + 0), \quad (18)$$

$$\varphi_m(r = a - 0) = \varphi_m(r = a + 0), \quad (19)$$

$$D_n(r = a - 0) = D_n(r = a + 0), \quad (20)$$

$$B_n(r = a - 0) = B_n(r = a + 0). \quad (21)$$

From the condition of electric scalar potential,  $\varphi$ , continuity, it is:

$$\frac{B_n}{a^{n+1}} = \left( C_n - \frac{q}{4\pi\epsilon_0 d^{n+1}} \right) a^n + \frac{q d^n}{4\pi\epsilon_0} \frac{1}{a^{n+1}}, \quad (22)$$

while from the magnetic scalar potential ( $\varphi_m$ ) continuity condition:

$$C_{1n} a^n = \frac{B_{1n}}{a^{n+1}}. \quad (23)$$

Furthermore, from the continuity condition of  $D_n$  on the boundary surface, where  $D_n$  is the normal component of vector  $\mathbf{D}$ , it is obtained:

$$-\varepsilon(n+1)\frac{B_n}{a^{n+2}} - \xi(n+1)\frac{B_{1n}}{a^{n+2}} = \varepsilon_0 n a^{n-1} \left( C_n - \frac{q}{4\pi\varepsilon_0 d^{n+1}} \right) - \varepsilon_0(n+1) \left( \frac{q d^n}{4\pi\varepsilon_0} \right) \frac{1}{a^{n+2}}. \quad (24)$$

Next, from the condition of vector  $B_n$  continuity on the boundary surface, where  $B_n$  is the normal component of vector  $\mathbf{B}$ , it can be obtained:

$$-\mu(n+1)\frac{B_{1n}}{a^{n+2}} - \xi(n+1)\frac{B_n}{a^{n+2}} = \mu_0 n a^{n-1} C_{1n}. \quad (25)$$

In the end, it is obtained:

$$C_n = \frac{q}{4\pi\varepsilon_0 d^{n+1}} + \frac{q d^n}{4\pi\varepsilon_0 a^{2n+1}} \frac{(n+1) \left\{ (\varepsilon_0 - \varepsilon) [\mu(n+1) + \mu_0 n] + \xi^2 (n+1)^2 \right\}}{\left\{ \varepsilon(n+1) + \varepsilon_0 n \right\} [\mu(n+1) + \mu_0 n] - \xi^2 (n+1)^2} \quad (26)$$

$$B_n = \frac{q d^n}{4\pi} \frac{(2n+1) [\mu(n+1) + \mu_0 n]}{\left\{ \varepsilon(n+1) + \varepsilon_0 n \right\} [\mu(n+1) + \mu_0 n] - \xi^2 (n+1)^2} \quad (27)$$

$$B_{1n} = -\xi \frac{q d^n}{4\pi} \frac{(2n+1)(n+1)}{\left\{ \varepsilon(n+1) + \varepsilon_0 n \right\} [\mu(n+1) + \mu_0 n] - \xi^2 (n+1)^2} \quad (28)$$

$$C_{1n} = -\xi \frac{q d^n}{4\pi a^{2n+1}} \frac{(2n+1)(n+1)}{\left\{ \varepsilon(n+1) + \varepsilon_0 n \right\} [\mu(n+1) + \mu_0 n] - \xi^2 (n+1)^2}. \quad (29)$$

### III. NUMERICAL RESULTS

It is of interest to consider parameters which appear in the constants  $C_n, B_n, C_{1n}, B_{1n}$  and which depend on  $\xi^2/\varepsilon_0\mu_0$ . Let's consider the case  $\varepsilon = 2\varepsilon_0$  and  $\mu = \mu_0$ . Furthermore, let's denoted parameters as:

$$p(\varepsilon/\varepsilon_0, \mu/\mu_0, n) = \frac{(n+1) \left\{ \left( 1 - \frac{\varepsilon}{\varepsilon_0} \right) \left[ (n+1) \frac{\mu}{\mu_0} + n \right] + \frac{\xi^2}{\varepsilon_0 \mu_0} (n+1)^2 \right\}}{\left[ \frac{\varepsilon}{\varepsilon_0} (n+1) + n \right] \left[ \frac{\mu}{\mu_0} (n+1) + n \right] - \frac{\xi^2}{\varepsilon_0 \mu_0} (n+1)^2} \quad (30)$$

$$p_1(\varepsilon/\varepsilon_0, \mu/\mu_0, n) = \frac{(2n+1)(n+1)}{\left[ \frac{\varepsilon}{\varepsilon_0} (n+1) + n \right] \left[ \frac{\mu}{\mu_0} (n+1) + n \right] - \frac{\xi^2}{\varepsilon_0 \mu_0} (n+1)^2} \quad (31)$$

$$\alpha(\varepsilon/\varepsilon_0, \mu/\mu_0, n) = \left[ \frac{\varepsilon}{\varepsilon_0} (n+1) + n \right] \left[ \frac{\mu}{\mu_0} (n+1) + n \right] - \frac{\xi^2}{\varepsilon_0 \mu_0} (n+1)^2 \quad (32)$$

and let us show their graphical dependence on bi-isotropy parameter  $\xi^2/(\varepsilon_0\mu_0)$ , when  $n$  is parameter.

From Fig. 2 and Fig.3 it can be seen that parameters  $p$  and  $p_1$  increase by increasing the bi-isotropy parameter  $\xi$ , whereas parameter  $\alpha$  decreases by increasing bi-isotropy parameter  $\xi$ .

In these three figures,  $n$  is the parameter.

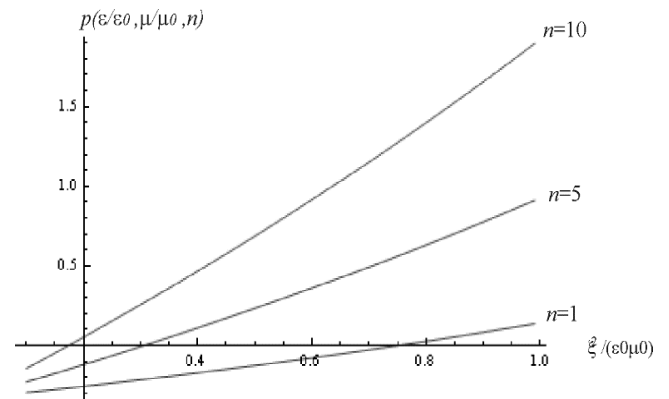


Fig. 2. Dependence  $p(\varepsilon/\varepsilon_0, \mu/\mu_0, n)$  on bi-isotropy parameter.

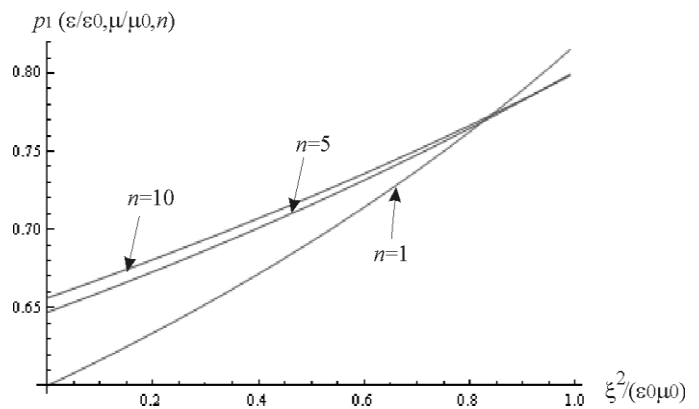


Fig. 3. Dependence  $p_1(\varepsilon/\varepsilon_0, \mu/\mu_0, n)$  on bi-isotropy parameter

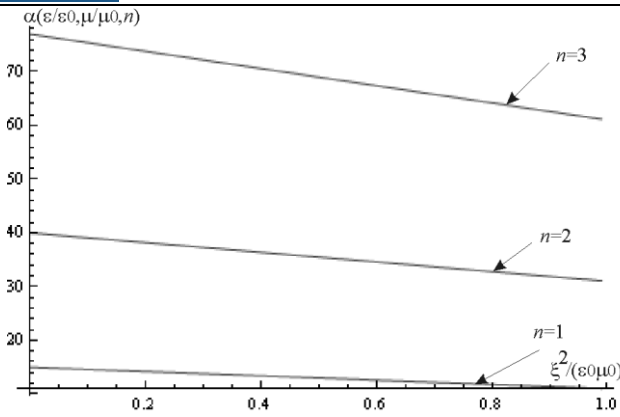


Fig. 4. Dependence  $\alpha(\varepsilon/\varepsilon_0, \mu/\mu_0, n)$  on bi-isotropy parameter  $\xi$ .

#### IV. CONCLUSION

In this paper, we have derived electric and magnetic scalar potential of point charge which is placed in air spherical hollow, which exists inside bi-isotropic medium of Tellegen type. The expressions are obtained by solving Legendre differential equations for electric and magnetic scalar potential and by satisfying boundary conditions for both potential continuity on boundary surfaces as well as continuity of normal components of vectors  $\mathbf{D}$  and  $\mathbf{B}$ .

As a consequence of specific constitutive relations for bi-isotropic mediums, constants  $C_n, B_n, C_{1n}, B_{1n}$  depend on bi-isotropy parameter, which characterizes the bi-isotropic properties of the surrounding environment outside air spherical hollow.

This further influences both the electrical and magnetic scalar potential, as well as the fields that exist in the vicinity of the point charge, which are obtained by equation (4). In addition, the presence of Tellegen's bi-isotropic environment, although the point charge is placed in the air hollow, leads to the appearance of a magnetic field. When  $\xi = 0$ , expressions are reduced to those that are derived [13] and magnetic field does not exist.

#### ACKNOWLEDGEMENT

This work was supported by the Serbian Ministry of Education, Science, and Technological Development under grant TR-32052 and III44004.

#### REFERENCES

- [1] A.H. Sihvola, "Electromagnetic modelling of bi-isotropic media", Progress In Electromagnetics Research, PIER no 9, pp.45-86,1994.
- [2] W. Ren, "Bi-isotropic layered mixtures", Progress in Electromagnetics research, PIER, 9, pp. 103-116, 1994.
- [3] A.H. Sihvola, A.J. Viitanen, I.V. Lindell, S.A. Tretyakov, *Electromagnetic Waves in Chiral and Bi-isotropic media*, (Artech House Antenna Library), ISBN-13: 978-0890066843.
- [4] I.V. Lindeli, "Quasi-static image theory for Bi-isotropic Sphere", IEEE Trans. Antennas and Propagat., Vol. 40, No.2, pp. 228-233, Feb.1992.
- [5] I. V. Lindell, J. J. Hänninen, "Static image principle for the sphere in isotropic or bi-isotropic space", Radio Science, Vol. 35, Issue 3, pp.653-660, 2000.
- [6] D. Veličković, Ž. Mančić, "Image theorem invariance in biisotropic media", PES 96, Niš, 22-23 May, pp. 59-63, 1996.
- [7] N. Raičević, S. Ilić. "One method for chiral and metamaterial strip lines determination", International conference in European Electromagnetics, EUROEM 2008, Lausanne, Switzerland, pp. 44, 21-25 July 2008.
- [8] D. Veličković, B. Milovanović, Z. Mančić, "New approach for equalising phase velocities in coupled microstrip lines", Electronic letters, Vol. 33, Issue 17, pp. 1444-1445, 1997.
- [9] Z. J. Mančić, V. V. Petrović, "Quasi-Static Analysis of the Shielded Microstrip Line with Bi-isotropic Substrate by the Strong FEM Formulation" (Proceedings Paper), 2013 11TH International Conference on Telecommunication in Modern Satellite, Cable and Broadcasting Services (TELSIKS) Vols 1 and 2, pp. 513-516, 2013.
- [10] Ž. J. Mančić, S. S. Ilić, "Contribution to the Electrostatic Analysis of Bi-isotropic Mediums of Tellegen Type", Proceedings of 13th International Conference on Telecommunication in Modern Satellite, Cable and Broadcasting Services (TELSIKS 2017), Niš, Serbia, October 18-20, pp. 359-362, 2017.
- [11] Ž. J. Mančić, Z. Ž. Cvetković, S. S. Ilić, "Contribution to the electrostatics of bi-isotropic materials in the case of lineic ring of charge placed beside a sphere of a bi-isotropic material", The Small Systems Simulation Symposium, CD P10, Niš, Serbia, pp. 59-62, 2018, ISBN 978-86-6125-199-3.
- [12] D. Veličković, "Wave equations and its solutions in bianisotropic media", Scientific meeting New sintered materials and their applications, Nis, pp. 40-41, 1990.
- [13] D. Veličković, *Calculation methods for electrostatic fields*, Stil, Podvis, Niš, 1982.