# An Original Approach To The Construction of $(3,2,\rho)$-N-Symmetrizable Hilbert Spaces

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***Abstract:*** **In this paper we will present an original approach to the construction of a special type of Hilbert space. Assuming that a set $M$ is given with a $(3,2,\rho)$-metric $d$ on it, the construction of $(3,2,\rho)$-N-symmetrizable Hilbert space will be described in detail. This Hilbert space is characterized by very interesting topological structure and properties.**

***Keywords*** **– $(3,2,\rho)$-metric, $(3,2,\rho)$-metric spaces, $(3,2,\rho)$-N-symmetrizable spaces**

## I. Introduction

The notion of metric space leads to an important topological notion, viz., the notion of a metrizable space. We lay great stress on metric and metrizable spaces because many important topological spaces used in various branches of mathematics are metrizable. Axiomatic classification and the generalization of metric and metrizable spaces have been considered in a lot of papers. We will mention some of them: K. Menger ([14]), V. Nemytzki, P.S. Aleksandrov ([16],[1]), Z. Mamuzic ([13]), S. Gähler ([12]), A. V. Arhangelskii, M. Choban, S. Nedev ([2],[3],[17]), J. Usan ([18]), B. C. Dhage, Z. Mustafa, B. Sims ([8], [15]). The notion of $(n,m,\rho)$-metric is introduced in [9]. Connections between some of the topologies induced by a $(3,1,\rho)$-metric $d$ and topologies induced by a pseudo-$o$-metric, $o$-metric and symmetric are given in [10]. For a given $(3,j,\rho)$-metric $d$ on a set $M$, $j\epsilon\{1,2\}$, seven topologies $\tau(G,d)$, $\tau(H,d)$, $\tau(D,d)$, $\tau(N,d)$, $\tau(W,d)$, $\tau(S,d)$ and $\tau(K,d)$ on $M$, induced by $d$, are defined in [4] and several properties of these topologies are shown, such as: $\tau(W,d)\subseteq\tau(N,d)\subseteq\tau(D,d)\subseteq\tau(G,d)$, $\tau(W,d)\subseteq\tau(H,d)\subseteq\tau(G,d)$, $\tau(W,d)\subseteq\tau(S,d)\subseteq\tau(K,d)$. For a $(3,2,\rho)$-metric $d$ on $M$, the following inclu-sions are also satisfied $\tau(W,d)\subseteq\tau(N,d)=\tau(S,d)=\tau(K,d)\subseteq\tau(D,d)\subseteq\tau(G,d)$.

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In this paper we consider only the topologies $\tau(N,d)$ and $\tau(D,d)$ induced by a $(3,2,\rho)$-metric $d$. We will prove that the set

$$\beta=\{B(a,a,\varepsilon)|a\in M,\varepsilon>0\}$$

is a base for $\tau(N,d)$ and the Hilbert space $H$ is a $(3,2,\rho)$-N-symmetrizable with topology $\tau=l^2(\text{on } R^3)$ and is a separable space.

## II. Some properties of $(3,2,\rho)$-N-symmetrizable spaces

In this part we state the notions (defined in [4]) used later.

Let $M$ be a nonempty set and let $d: M^3\to \mathrm{R}_0^+=[0,\infty)$.

We state five conditions for such a map.

$(\boldsymbol{M0})$ $d(x,x,x)=0$, for any $x\in M$;

$(\boldsymbol{P})$ $d(x,y,z)=d(x,z,y)=d(y,x,z)$, for any $x,y,z\in M$

$(\boldsymbol{M1})$ $d(x,y,z)\leq d(x,y,a)+d(x,a,z)+d(a,y,z)$, for any $x,y,z,a\in M$;

$(\boldsymbol{M2})$ $d(x,y,z)\leq d(x,a,b)+d(a,y,b)+d(a,b,z)$, for any $x,y,z,a,b\in M$;

$(\boldsymbol{Ms})$ $d(x,x,y)=d(x,y,y)$, for any $x,y\in M$.

For a map $d$ as above let

$\rho=\{(x,y,z)|(x,y,z)\in M^3, d(x,y,z)=0\}$.

The set $\rho$ is a $(3,j)$-equivalence on $M$, as defined and discussed in [9], [4].

The set $\Delta=\{(x,x,x)|x\in M\}$ is (3,1)-equivalence on $M, j=1,2$, and the set $\nabla=\{(x,x,y)\mid x,y\in M\}$ is a (3,1)-equivalence, but it is not a (3,2)-equivalence on $M$. However, the condition $(\boldsymbol{M0})$ implies that $\Delta\subseteq\rho$.

**Definition.2.1.** Let $d: M^3\to \mathrm{R}_0^+$ and $\rho$ be as above. If $d$ satisfies $(\boldsymbol{M0})$, $(\boldsymbol{P})$ and $(\boldsymbol{Mj})$, $j\in\{1,2\}$, we say that $d$ is a $(3,j,\rho)$**-metric** on $M$. If $d$ is a $(3,j,\Delta)$-metric on $M$, we say that $d$ is a $(\boldsymbol{3},\boldsymbol{j})$**-metric** on $M$. If $d$ is a $(\boldsymbol{3},\boldsymbol{j},\boldsymbol{\rho})$**-metric** and satisfies $(\boldsymbol{Ms})$, we say that $d$ is a $(\boldsymbol{3},\boldsymbol{j},\boldsymbol{\rho})$**-symmetric** on $M$, and if $d$ is a $(\boldsymbol{3},\boldsymbol{\rho})$**-metric** and satisfies $(\boldsymbol{Ms})$, we say that d is a $(\boldsymbol{3},\boldsymbol{\rho})$**-symmetric** on M.

**Remark 2.1.** Any $(3,j,\rho)$-metric $d$ on $M$ induces a map $D_d: M^2\to R_0^+$ defined by:

$$D_d(x,y)=d(x,x,y).$$

It is easy to check the following facts.

***a)*** For any $(3,j,\rho)$-metric $d$, $D_d(x,x)=0$. ($D_d$ is called a **distance** in [15] and a **pseudo o-metric** in [17].)

***b***) For any $(3, j)$-metric $d$, $D_d(x, y) = 0$ if and only if $x = y$. ($D_d$ is called an ***o*-metric** in [17].)

***c***) For any $(3, j)$-symmetric $d$, $D_d(x, y) = D_d\ (y, x)$. ($D_d$ is called a **symmetric** in [17].)

***d***) For any $(3,2, \rho)$-metric $d$,
$D_d(x, y) \lesssim 2D_d(z, x) + D_d(z, y)$ and
$D_d(x, y) \lesssim 2D_d(y, x)$.

***e***) For any (3,2)-symmetric $d$,
$D_d(x, y) = D_d(y, x) \lesssim 3(D_d(x, z) + D_d(z, y))/2$. (In the literature $D_d$ is called a **quasimetric,** a **nearmetrics** or an **inframetrics**.)

Let $d$ be a $(3,2, \rho)$-metric on $M$, $x, y \in M$ and $\varepsilon > 0$. As in [4], we consider the following $\varepsilon$-ball, as subset of $M$

$B(x, y, \varepsilon) = \{z | z \in M, d(x, y, z) < \varepsilon\}$–$\varepsilon$-ball
with center at $(x, y)$ and radius $\varepsilon$.

**Remark 2.2. a)** For $x = y$, $B(x, x, \varepsilon) = B(x, y, \varepsilon) = \{z \mid z \in M, d(x, x, z) < \varepsilon\}$. For any $a \in M$, $a \in B(a, a, \varepsilon)$, but, it is possible for some $x \neq a$ to have $a \notin B(a, x, \varepsilon)$.

***b***) For a pseudo $o$-metric $D: M^2 \to R_0^+$, there is only one possibility for defining $\varepsilon$ -balls, i.e.

$B(x, \varepsilon) = \{z \mid z \in M, D(x, z) < \varepsilon\}$.

Among the others, a $(3,2, \rho)$-metric $d$ on $M$ induces the following two topologies as in [4]:

**1**) $\tau(N, d)$–the topology defined by: $U \in \tau(N, d)$ iff $\forall x \in U$, $\exists \varepsilon > 0$ such that $B(x, x, \varepsilon) \subseteq U$;

**2**) $\tau(D, d)$ –the topology generated by all the $\varepsilon$-balls $B(x, x, \varepsilon)$.

In [4] we proved that $\tau(N, d) \subseteq \tau(D, d)$ for any $(3,2, \rho)$-metric $d$. With the next proposition we will show that $\tau(N, d) = \tau(D, d)$ for any $(3,2, \rho)$-symmetric $d$.

**Proposition 2.1.** For any $(3,2, \rho)$-symmetric $d$ on $M$, the ball $B(a, a, \varepsilon) \in \tau(N, d)$, for any $a$ on $M$ and $\varepsilon > 0$.

**Proof:** It is enough to show that for any $x \in B(a, a, \varepsilon)$ there is $\delta > 0$, such that $B(x, x, \delta) \subseteq B(a, a, \varepsilon)$. Let $x \in B(a, a, \varepsilon)$ and $\delta = (\varepsilon - D_d(a, x))/2$. Then, for any $z \in B(x, x, \delta)$ we have that

$$
\begin{aligned}
D_d(a, z) &= D_d(z, a) \\
&\leq 2D_d(z, x) + D_d(x, a) \\
&< 2\delta + D_d(x, a) \\
&= 2(\varepsilon - D_d(a, x))/2 + D_d(x, a) = \varepsilon.
\end{aligned}
$$

This implies that $z \in B(a, a, \varepsilon)$, i.e., $z \in B(x, x, \delta) \subseteq B(a, \varepsilon)$.

This proposition shows that the set $\beta = \{B(a, a, \varepsilon) | a \in M, \varepsilon > 0\}$ is a base for $\tau(N, d)$. Moreover, $\tau(N, d) = \tau(D, d)$.

**Definition 2.2.** We say that a topological space $(M, \tau)$ is $(3,2, \rho)$-N-symmetrizable via a $(3,2, \rho)$-symmetric $d$ on $M$, if $\tau = \tau(N, d)$.

## III. Construction of $(3,2, \rho)$-N-symmetrizable Hilbert space

In [4] we proved that any (3,2)-N-D-metrizanle spaces $(M, \tau)$ is metrizable, so by this in [6] we proved that any (3,2)-N-metrizable spaces $(M, \tau)$ is perfectly normal. The last property of the perfect normality of the (3,2)-N-metrizable space allows us to make the constructed Hilbert space to be metrizable if $\rho = \Delta$, even more so that it is separable. First, we construct the space.

**Proposition 3.1.** The Hilbert's space is $(3,2, \rho)$-N-symmetrizable space via a $(3,2, \rho)$-symmetric $d$.

**Proof:** Let $M = H$ be the set of all infinite sequences $\{x_i\}_{i=1}^{\infty}$ of real numbers satisfying the condition $\sum_{i=1}^{\infty} x_i^2 < \infty$ and let $d(x, x, y) = D_d(x, y)$, where $D_d$ is defined in Remark 2.2.

We shall show that with

$$D_d(x, y) = \sqrt{\sum_{i=1}^{\infty}(x_i - y_i)^2},$$

for $x = \{x_i\}_{i=1}^{\infty}$, $y = \{y_i\}_{i=1}^{\infty}$, a $(3,2, \rho)$-symmetric $d$ is defined.

First of all, we will prove that $D_d$ is well-defined, i.e., that the series in the definition of $D_d$ is convergent. In the proof we shall apply the Cauchy inequality

$$\left|\sum_{i=1}^{k} a_i b_i\right| \leq \sqrt{\sum_{i=1}^{k} a_i^2} \cdot \sqrt{\sum_{i=1}^{k} b_i^2},$$

that holds for all finite sequences $a_1, a_2, \cdots, a_k$ and $b_1, b_2, \cdots, b_k$ of real numbers.

Let us note that for every pair of points $x = \{x_i\}_{i=1}^{\infty}$, $y = \{y_i\}_{i=1}^{\infty}$ in $H$ and any positive integer $k$ we have

$$
\begin{aligned}
&\sum_{i=1}^{k}(x_i - y_i)^2 = \sum_{i=1}^{k} x_i^2 - 2\sum_{i=1}^{k} x_i y_i + \sum_{i=1}^{k} y_i^2 \\
&\leq \sum_{i=1}^{k} x_i^2 + 2\sqrt{\sum_{i=1}^{k} x_i^2} \cdot \sqrt{\sum_{i=1}^{k} y_i^2} + \sum_{i=1}^{k} y_i^2 \\
&= \left(\sqrt{\sum_{i=1}^{k} x_i^2} + \sqrt{\sum_{i=1}^{k} y_i^2}\right)^2 \\
&\leq \left(\sqrt{\sum_{i=1}^{\infty} x_i^2} + \sqrt{\sum_{i=1}^{\infty} y_i^2}\right)^2.
\end{aligned}
$$

Since the last inequality holds for any positive integer $k$, the series in the definition of $D_d$ is convergent and $D_d(x, y)$ is well-defined.

Obviously, $D_d$ satisfies conditions $(\boldsymbol{M0})$, $(\boldsymbol{P})$ and $(\boldsymbol{Ms})$. In addition, we shall show that condition $(\boldsymbol{M2})$ is also satisfied.

Let $x = \{x_i\}_{i=1}^{\infty}$, $y = \{y_i\}_{i=1}^{\infty}$ and $z = \{z_i\}_{i=1}^{\infty}$ be any points of $H$, let

$$
\begin{aligned}
x^k &= \{x_1, x_2, \cdots, x_k, 0,0, \cdots\}, \\
y^k &= \{y_1, y_2, \cdots, y_k, 0,0, \cdots\}, \\
z^k &= \{z_1, z_2, \cdots, z_k, 0,0, \cdots\}
\end{aligned}
$$

and

$$a_i = x_i - y_i, \quad b_i = y_i - z_i, \quad c_i = x_i - z_i.$$

By the Cauchy inequality we have

$$
\begin{aligned}
&|D_d(x^k, z^k)|^2 = |D_d(z^k, x^k)|^2 \\
&= \sum_{i=1}^{k} c_i^2 = \sum_{i=1}^{k}(a_i + b_i)^2 \\
&= \sum_{i=1}^{k} a_i^2 + 2\sum_{i=1}^{k} a_i b_i + \sum_{i=1}^{k} b_i^2 \\
&\leq \sum_{i=1}^{k} a_i^2 + 2\sqrt{\sum_{i=1}^{k} a_i^2} \cdot \sqrt{\sum_{i=1}^{k} b_i^2} + \sum_{i=1}^{k} b_i^2 = \\
&\left(\sqrt{\sum_{i=1}^{k} a_i^2} + \sqrt{\sum_{i=1}^{k} b_i^2}\right)^2 \\
&< 9\left(\sqrt{\sum_{i=1}^{k} a_i^2} + \sqrt{\sum_{i=1}^{k} b_i^2}\right)^2 /4 \\
&= 9|D_d(x^k, y^k) + D_d(y^k, z^k)|^2/4.
\end{aligned}
$$

From the last inequality it follows that for $k = 1,2,\ldots$ we have

$$3(D_d(x,y) + D_d(y,z))/2 \geq 3(D_d(x^k,y^k) + D_d(y^k,z^k))/2 \geq D_d(x^k,z^k),$$

and this implies that

$$3(D_d(x,y) + D_d(y,z))/2 \geq D_d(x,z).$$

From the last inequality we have that $D_d(x,y) = \sqrt{\sum_{i=1}^{\infty}(x_i - y_i)^2}$, defines $(3,2,\rho)$-symmetric on $H$. We also have the topology $\tau$, induced by the metric $d$, in literature known as $l^2$-topology so the topological space $(H,\tau)$ where $\tau = l^2$, is a Hilbert $(3,2,\rho)$-N-symmetrizable space via a $(3,2,\rho)$-symmetric $d$. Moreover, if $\rho = \Delta$ the space $(H,\tau)$ is metrizable. ■

The set of all sequences $\{x_i\}_{i=1}^{\infty}$, where all the $x_i$'s are rational numbers only finitely many of which are distinct from zero, is dense in $(H,\tau)$ and countable, so the Hilbert's $(3,2,\rho)$-N-symmetrizable space is also separable.

## IV. Conclusion

Under the assumption that on the set $H$ a $(3,2,\rho)$- metric $d$ is defined, we considered the induced topologies $\tau(N,d)$ and $\tau(D,d)$. Due to the symmetric properties of metric $d$, we proved that these topologies are equal. In addition, we constructed the $(3,2,\rho)$-N-symmetrizable space Hilbert space, by showing that conditions mentioned in II are satisfied. Moreover, this space is a metrizable and also a separable Hilbert space.

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