On Groups of Cohomologies on a Locally Compact Topological Space

Elena Kotevska[1] and Sonja Chalamani[1]

Abstract – **In this paper we will present an example of a group of cohomologies on a locally compact topological space. Moreover, the properties of this group of cohomologies will be examined and proved in detail.**

Keywords – **Topological space, compact set, $p-$measurable function, $p-$measurable coboundary, group of cohomologies.**

I. INTRODUCTION

Let X be a topological space, and let G be an Abelian group. In [1] we introduced the Abelian group $\Phi^p(X,G)$ of p-functions and we constructed the homeomorphism

$$\delta\colon \Phi^p(X,G) \to \Phi^{p+1}(X,G).$$

This homomorphism is defined for every $p \geq 0$ such that for $\varphi \in \Phi^p(X,G)$ the corresponding $\delta\varphi \in \Phi^{p+1}(X,G)$ is given by

$$(\delta\varphi)(x_0,x_1,\cdots,x_p,x_{p+1}) = \sum_{i=0}^{p+1}(-1)^i\varphi(x_0,x_1,\cdots,x_{i-1},\hat{x}_i,x_{i+1},\cdots,x_{p+1}),$$

where "^" signs that we dropped the marked coordinate. Moreover, we proved the following properties

(1) $\Phi_0^p(X,G) \subset \Phi_C^p(X,G)$
(2) $\Phi_0^p(X,G), \Phi_C^p(X,G)$
(3) $\delta\left(\Phi_0^p(X,G)\right) \subseteq \Phi_0^{p+1}(X,G)$
(4) $\delta\left(\Phi_C^p(X,G)\right) \subseteq \Phi_C^{p+1}(X,G)$,

for the sets

$$\Phi_0^p(X,G) = \{\varphi|\varphi \in \Phi^p(X,G), |\varphi| = \emptyset\} \text{ and}$$
$$\Phi_C^p(X,G) = \{\varphi|\varphi \in \Phi^p(X,G), |\varphi| = compact\}$$

and

$$\Phi_0^p(X,G) = \{\varphi|\varphi \in \Phi^p(X,G), |\varphi| = \emptyset\},$$

where $|\varphi|$ is the ***support of p-function***.

Lastly, we proved that the quotient spaces

$$C_C^p(X,G) = {\Phi_C^p(X,G)}/{\Phi_0^p(X,G)} \text{ and}$$
$$C^p(X,G) = {\Phi^p(X,G)}/{\Phi_0^p(X,G)}$$

inherit the properties of the Abelian mapping that a homeomorphism from the Abelian group $\Phi^p(X,G)$.

II. DEFINITION OF GROUPS OF COHOMOLOGIES

We define the sets

$$Z^p(X,G) = \{u|u \in C_C^p(X,G), \delta u = 0\}$$

$$B^p(X,G) = \delta[C_C^{p-1}(X,G)].$$

Since δ is a homomorphism between groups, it immediately follows that the sets $Z^p(X,G)$ and $B^p(X,G)$ are subgroups of the group $C_C^p(X,G)$. Moreover, these sets are actually the Kernel and the Image of the corresponding homomorphisms δ. They are called groups of **$p-$measurable cocycles** and **$p-$measurable coboundaries** (of topological space X with coefficients in the group G) correspondingly. It can be shown that for a homomorphism δ, $\delta \circ \delta=0$ is valid ([1]). By this we immediately have that the group of $p-$measurable coboundaries $B^p(X,G)$ is a subgroup of the group of $p-$measurable cocycles $Z^p(X,G)$. This means that we can define the quotient group

$$H_C^p(X,G) = {Z^p(X,G)}/{B^p(X,G)}.$$

This group is called **$p-$measurable group of cohomologies** with compact supports on topological space X, with coefficients in the group G. In this paper, we envisage the groups $H_C^p(X,G)$, for a locally compact topological space X. If we consider a compact topological space, then the group of cohomologies is denoted by $H_C^p(X,G)$. In such case, we also write $\Phi^p(X,G) = \Phi_C^p(X,G)$.

It is a general impression, at least to this level of presentation of definitions, that they are somehow unnatural and just scattered without any real meaning. However, as one digs deeper into the material, one discovers the importance and applications of theory of cohomologies, as well as it's not so apparent naturalness of the abovementioned definitions. Groups of cohomologies can be introduced, i.e., defined in lots

[1]Elena Kotevska and Sonja Chalamani are with the Faculty of Technical Sciences at the University st. Kliment Ohridski, 7000 Bitola, Republic of Macedonia,
E-mail: elena.kotevska@tfb.uklo.edu.mk
scalamani@yahoo.com

of different ways. For compact topological spaces with „good" local behavior, different definitions lead to the same result. For „pathological" spaces however, different definitions can lead to the different results. Considering that each definition is based on a set of different assumptions they all have some advantages or/and drawbacks with respect to the other. In this presentation, we have chosen the approach that has the advantage of being general and yet simple. By this approach, we arrive very quickly to the subject of interest and study. In the following, we will present a very simple example of group of cohomologies.

III. An Example of Groups of Cohomologies

We notice that for any topological space X and any $p < 0$

$$H_C^p(X,G) = 0.$$

Proposition 1: Let the topological space X consist of exactly one point, i.e., let $X = \{a\}$. Then

$$H_C^p(X,G) = 0, p > 0$$

and

$$H_C^0(X,G) \cong G.$$

Lemma 1: Let the topological space X consist of exactly one point, i.e., let $X = \{a\}$. Then

1. $\Phi^p\,(X,G) \cong G$
2. $\Phi_0^p(X,G) = \{0\}$
3. $C_C^p(X,G) = C^p\,(X,G) = \Phi^p\,(X,G)$

Proof:

1. Let $p \geq 0$ be any given, but fixed natural number. Then the set of all $p+1$ $-$taples $(x_0, x_1, \cdots, x_p)$, where $x_i \in X$ and $i = 0, \cdots, p$ has exactly one element $(a, a, \cdots, a)$ with a showing p+1 times. We can conclude that X^p is a one-element set. So, for any $p \geq 0$ and any $p -$ function φ we can consider that

$$\varphi(x_0, x_1, \cdots, x_p) = s \in G,$$

i.e., we can say that every element $\varphi \in \Phi^p\,(X,G)$ maps the $p+1$ $-$tupple $(a, a, \cdots, a)$ to an unique element $s \in G$. This way every element $\varphi \in \Phi^p\,(X,G)$ for any $p \geq 0$ determines a unique element of the group G. The opposite is also valid, i.e., each element of $s \in G$ determines a unique element $\varphi \in \Phi^p\,(X,G)$. This means that we have a bijection between groups $\Phi^p\,(X,G)$ and G given by $f: G \to \Phi^p\,(X,G)$

$$f(s) = \varphi \Leftrightarrow \varphi(a, a, \cdots, a) = s$$

We set $\varphi \equiv \varphi_s$. The mapping f is also a homomorphism. Indeed, for any $s_1, s_2 \in G$ we have

$$\begin{aligned} f(s_1+s_2)(a,a,\dots,a) &= \varphi_{s_1+s_2}(a,a,\cdots,a) = s_1 + s_2 \\ &= \varphi_{s_1}(a,a,\cdots,a)+\varphi_{s_2}(a,a,\cdots,a) \\ &= (f(s_1)+f(s_2))(a,a,\dots,a). \end{aligned}$$

This yields that the mapping f is an isomorphism, q.e.d.

2. Let $\varphi \in \Phi_0^p(X,G)$. Then the support $|\varphi| = \emptyset$ and for the element $a \in X$, $a \notin |\varphi|$. This means that there is a neighborhood V of a, $a \in V$, such that for every $(x_0, x_1, \cdots, x_p) \in V$, $\varphi(x_0, x_1, \cdots, x_p) = 0$. Actually, we have $\varphi(a, a, \cdots, a) = 0$ with $X = \{a\}$. It follows that $\varphi = \mathbf{0}$. This shows that $\Phi_0^p(X,G) \subseteq \{0\}$. The opposite is always valid, so it follows that $\Phi_0^p(X,G) = \{0\}$.

3. It follows from 2, that

$$C^p\,(X,G) = {\Phi^p\,(X,G)}\Big/{\{0\}} = \Phi^p\,(X,G) \cong G.$$

On the other hand, $\Phi_C^p(X,G) = \Phi^p\,(X,G)$, since a topological space of one point is always compact. It follows that

$$C_C^p(X,G) = {\Phi_C^p(X,G)}\Big/{\{0\}} = \Phi^p\,(X,G) \cong G, \text{ q.e.d.}$$

We now proceed with the prove of the Proposition.

Let $p > 0$ be any natural number. Then, for any $p -$ function $\varphi \in \Phi^p\,(X,G)$, we have

$$(\delta(\varphi))(a, a \dots, a) = \sum_{i=0}^{p+1} (-1)^i \varphi(a, a, \dots, a, \hat{a}, a \dots, a),$$

where $\hat{a}$ is on position with index i.

We will consider two cases:

1. Let $p > 0$ be an even number. Then, $p+1$ is odd and on the right side we have even number of additions and so the sum will be zero. This means that

$$(\delta(\varphi))(a, a \dots, a) = 0, \text{ i.e., } \delta(\varphi) = \mathbf{0}.$$

2. Let $p > 0$ be an odd number. Then, $p+1$ is even and on the right side we have odd number of additions. This means that

$$(\delta(\varphi))(a, a \dots, a) = -\varphi(a, a, \dots, a),$$

where on the left we have $p+1$ $-$tupples and on the right we have p $-$tupples. The nature of the groups considered gives us

$$\delta(\varphi) = -\varphi,$$

because $\Phi_0^p(X,G) = \{0\}$ and $C_C^p(X,G) = \Phi^p\,(X,G)$.

As a result of the above we get:

1. For even $p > 0$

$\delta: C_C^p(X,G) \to C_C^{p+1}(X,G)$ is such that $\delta(\varphi) = \mathbf{0}$, for every $\varphi \in C_C^p(X,G)$ and so we have that

$$Z^p\,(X,G) = C_C^p(X,G).$$

On the other side $p-1$ is odd and thus we have $\delta(\varphi) = -\varphi$, for every $\varphi \in C_C^{p+1}(X,G)$. It follows that

$$\delta\left[\, C_C^{p-1}(X,G)\right] = C_C^p(X,G),$$

i.e.,

$$B^p\,(X,G) = C_C^p(X,G).$$

Consequently we get

$$H_C^p(X,G) = {Z^p\,(X,G)}\big/{B^p\,(X,G)} = {C_C^p(X,G)}\big/{C_C^p(X,G)} = \{\mathbf{0}\}.$$

2. For odd $p > 0$, we have that for every $\varphi \in C_C^p(X,G)$, $\delta(\varphi) = -\varphi$, i.e., we have

$$Z^p\,(X,G) = \left\{\varphi \in C_C^p(X,G), \middle|\, \delta(\varphi) = \mathbf{0}\right\} = \{\mathbf{0}\}.$$

On the other side $p-1$ is even and thus we have $\delta(\varphi) = 0$, for every $\varphi \in C_C^{p-1}(X,G)$. It follows that

$$B^p\,(X,G) = \delta\left[C_C^{p-1}(X,G)\right] = \{\mathbf{0}\}.$$

It follows that

$$H_C^p(X,G) = {Z^p\,(X,G)}\big/{B^p\,(X,G)} = {\{\mathbf{0}\}}\big/{\{\mathbf{0}\}} = \{\mathbf{0}\}.$$

If $p = 0$, then since it is considered an even number, for

$$\delta: C_C^0(X,G) \to C_C^1(X,G)$$

we have

$$Z^0\,(X,G) = C_C^0(X,G) = G$$

and

$$B^0\,(X,G) = \{\mathbf{0}\}.$$

It follows that

$$H_C^0(X,G) = {Z^0\,(X,G)}\big/{B^0\,(X,G)} = {G}\big/{\{\mathbf{0}\}} = G.$$

This proves the Proposition 1.

IV. Conclusion

Following the definitions, structure and the general approach in [1], we presented a way to define groups of cohomologies. Then, in order to give life to these definitions, we've presented an example of such a group of cohomology. For our example we chosed a simple compact topological space of only one point. Following the sets and definitions we constructed the group of cohomologies $H_C^p(X,G)$ for this topological space. Moreover, we have given exact calculation of the coboundary operator δ. By doing this we are trying to give "life" to seemingly sterile and unnatural constructions.

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