Piecewise Uniform Product Two-Dimensional Laplace Source Quantizatoin

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Abstract – **In this paper simple and complete asymptotical analysis is given for a piecewise uniform product two-dimensional Laplace source quantizer (PUPTDLSQ). PUPTDLSQ is based on uniform product two-dimensional Laplace source quantizers. Product quantizer optimality conditions and all main equation for a number of phase divisions and optimal number of levels for each partition are presented. These systems, may have asymptotic performance arbitrarily close to the optimum. Futher more, their anallysis and implementation can be simpler than those of optimal systems.**

Keywords – **Laplace source, quantization, optimal distorsion.**

I. Introduction

It has been that vector quantization has much of the priority in speech and image coding application over scalar quantization. Many studies have considered the design of the suboptimal polar vector quantizer. These schemes should have provided better performances than those of the rectangular-coordinate-based quantizers, but with simpler implementation than optimal scalar quantizers. In this paper optimization of two-dimensional Laplace source quantization is examined and existence of a single minimum in dependence of the number of points on levels is proven . The procedure for optimizing decision levels, representation levels, and number of points per levels with a constraint on the total number of points is given.

The optimal vector quantizer are usually constrained by low rates. The Laplace source quantization at high rates has been analyzed in [1] and [2]. Here we present a quantizer similar to the iterative polar quantizer described in [3] and [4].

The probability density function for independent identically distributed Laplace source with the zero mean value and the unity varience is given as

$$f(x) = \frac{1}{2} e^{-\sqrt{2}(|x_1|+|x_2|)} \tag{1}$$

x is the source vector with element x_1 and x_2. In order to simplify the vector quantizer design we. introduce a transformation of variables.

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The transformation is defined

$$r = \frac{1}{\sqrt{2}}(|x_1| + |x_2|) \tag{2}$$

as:

$$u = \frac{1}{\sqrt{2}}(|x_1| - |x_2|) \tag{3}$$

As a performance criterion for our discussion we consider the mean-squared error per dimension (MSE) which has experienced wide application due to its tractability and interpretation as quantization noise power. In the case of a two-dimensional source, it is shown that it gives the best result in the field of the implementation. Mean squared error is given as:

$$mse = \int_{R^k} |x - Q(x)|^2 f(x)dx = \sum_{i=1}^{N} \int_{R_i} |x - y_i|^2 f(x)dx \tag{4}$$

In this paper we consider quantizer similar to a uniform magnitude quantizer but we allow different number of quantization points at different magnitude levels. We want to optimaze the number of points at each level for this type of quantizer.

The obtained probability density function is:

$$f(r,u) = \frac{1}{2} e^{-2r} \tag{5}$$

The support region for scalar quantizers has been found in [5-6] by minimization of the total distortion D, which is a combination of granular (D_g) and overload (D_o) distortions, $D = D_g + D_o$.In paper [7] only granular distortion was examined and although, arragement of points N_i in L partitions was defined, type of cells and their arrangement within partitions wasn't considered. Paper [8] is an anex of paper [7], but the imperfection of this paper lies in using cubic cells for partitions and subpartitions. Due to this fact, optimal arrangement of points in a partition can't be found. The goal of this paper is to solve quantization problem in a case of PUPTDLSQ and to find corresponding support region. It is done by analytical optimization of the granular distortion and numerical optimization of the total distortion. We improve the cell size and use more optimal cell division in each partition. More precisely, our quantizer divides the input plane into L regions and every region is further subdivided into L_i ($1 \le i \le L$) subregions. I-th partition in the signal plane is allowed to have M_i ($1 \le i \le L$) cells in the phase quantizer. We perform two-steps optimization: 1) distortion optimization ($D(i)$) in every partition under the constraint

$4L_i x M_i = N_i$ and 2) optimization of the total granular distortion $D_g = \sum_{i=1}^{L} D(i)$ which achieves the optimal number of points N_i on each subpartition under the constraint $\sum_{i=1}^{L} N_i = N$. We give the construction procedure and present the Laplace source example.

II DESCRIPTION AND OPTIMIZATION

We define an N-points product quantizer *Q (LxM)* as a mapping $Q: R^2 \to C$ where R^2 is the real two dimensional space and

$$C \equiv \{ y_1 = (m_1, \psi_{1,1}), \cdots, y_N = (m_L, \psi_{L,M}) \} \tag{6}$$

is the output set or codebook with size $|C| = N$. The output vectors, $\mathbf{y}_i$ are sometimes refered to as output points, or reproduction vectors. Associated with every N point quantizer is a partition of the real space R^2 into N cells R_i, for i=1,…,N. The i-th cell is given by $R_i = \{x \in R^2 : Q(x) = y_i\}$, which is inverse image of y_i under Q. From this definition it follows that $\cup_i R_i = R^2$ and $R_i \cap R_j = 0$ for $i \neq j$. A cell that is unbounded is called an overload cell. Each bounded cell is called a granular cell. Together all of overload (granular) cells are called the overload region (granular region). The nonlinear compressor characteristic is used in paper [4]. Although the smooth and differentiable compressor characteristic is convenient for mathematical manipulations, there are problems of accurately implementing analog nonlinearities [11]. Today's technology allows uniform quantizers or piecewise linear compressor characteristics implementation. PUPTDLSQ can approximate smooth curves of the nonlinear compressor characteristics. A piecewise uniform quantizer range consists of several segments, each containing several quantization cells and output points corresponding to a uniform quantizer. Different segments, however, may have different step-sizes. In this paper, we give the simplest piecewise uniform product quantization and show that it has approximately same performances as NPQ but it's much simpler for application.

Let consider PUPTDLSQ of *L* partitions, each partition containing L_i subpartitions. In order to minimize the total distortion we proceed as follows: magnitude partition decision levels and reconstruction subpartition levels are given as

$$\begin{gathered} r_i = (i-1)\Delta ; 1 \le i \le L; \quad r_{L+1} = r_{\max}; \\ r_{i,j} = r_i + (j-1)\Delta / L_i ; 1 \le i \le L, \quad 1 \le j \le L_i + 1; \\ r_{L,L_{L+1}} = r_{L+1} = r_{\max}; \\ m_{i,j} = r_i + (j - 1/2)\Delta / L_i ; \quad 1 \le i \le L, \quad 1 \le j \le L_i \end{gathered} \tag{7}$$

where $\Delta = r_{\max} / L$.

The distortion is a sum of granular and overload distortion $D = D_g + D_o$:

$$\begin{gathered} D = \sum_{i=1}^{L} D(i) + D_o = \\ \{\frac{1}{2}\sum_{i=1}^{L}\sum_{j=1}^{L_i}\sum_{k=1}^{M_i} \int_{r_{i,j}}^{r_{i,j+1}} \int_{u_{i,j,k}}^{u_{i,j,k+1}} [(r - m_{i,j})^2 + (u - \hat{u}_{i,j,k})^2] \frac{1}{2} e^{-2r} \, dr du\} \\ + \{2m_{L,L_L} \int_{r_{\max}}^{\infty} (r - m_{L,L_L})^2 e^{-2r} dr + \frac{2}{3} \frac{m_{L,L_L}{}^3}{M_L{}^2} \int_{r_{\max}}^{\infty} e^{-2r} dr\} \end{gathered} \tag{8}$$

After integration over u and the reordering, D_g becomes

$$D_g = \sum_{i=1}^{L}\sum_{j=1}^{L_i} \left(2m_{i,j} \frac{\Delta_i^2}{12} + \frac{2m_{i,j}^3}{3M_i^2} \right) P_r(m_{i,j}) \tag{9}$$

where

$$\Delta_i = \frac{\Delta}{L_i} \text{ and } P_r(m_{i,j}) = 2e^{-2m_{i,j}} \Delta_i . \tag{10}$$

After the reordering of sum and integration over r it obtains

$$D(i) = \frac{1}{2}\left(\frac{\Delta_i^2}{12} P_i + \frac{2}{3M_i^2} I_i\right) \tag{11}$$

where

$$P_i = 4\int_{r_i}^{r_{i+1}} r e^{-2r} dr , I_i = 2\int_{r_i}^{r_{i+1}} r^3 e^{-2r} dr , M_i = \frac{N_i}{4L_i} \tag{12}$$

and

$$D_g = \sum_{i=1}^{L} D(i). \tag{13}$$

The i-th partition distortion is

$$D_i = \frac{1}{2}\left(\frac{1}{12}\frac{\Delta^2}{L_i^2} P_i + \frac{32L_i^2}{3N_i^2} I_i\right) \tag{14}$$

After solving

$$\frac{\partial D_i}{\partial L_i} = 0 \tag{15}$$

we obtain

$$L_{iopt} = \sqrt[4]{\frac{\Delta^2 P_i N_i^2}{128 I_i}} \tag{16}$$

and

$$M_i = N_i / 4L_{iopt} = \sqrt[4]{\frac{I_i N_i^2}{2\Delta^2 P_i}} . \tag{17}$$

Optimal solution is found applying the method of Lagrange multipliers.

$$J = \sum_{i=1}^{L} D(i) + \lambda \sum_{i=1}^{L} N_i , \tag{18}$$

$$\frac{\partial J}{\partial N_i} = 0 \tag{19}$$

yielding :

$$N_i = N \frac{\sqrt[4]{P_i I_i}}{\sum_{j=1}^{L} \sqrt[4]{P_j I_j}} \quad \text{for fixed } N. \tag{20}$$

The final expression for D_g is:

$$D_g = \frac{2\sqrt{2}\Delta}{3N} \left(\sum_{i=1}^{L} \sqrt[4]{P_i I_i} \right)^2 \tag{21}$$

III NUMERICAL ANALYSIS AND RESULTS

As an illustration of the PUPTDLSQ performance, we show the gain $G(dB) = 10\log(D_g^{2D} / D_{gskal})$ as a function of the number of bits per sample R.

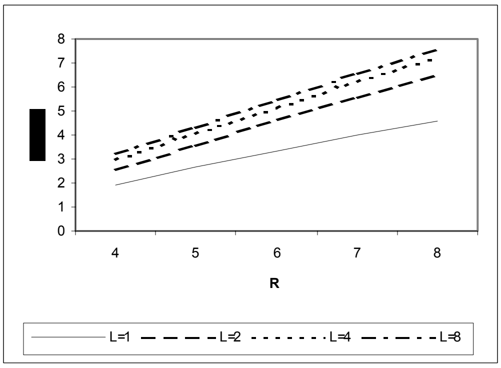

Fig. 1. Gain as a function of number of bits per sample

In order to see advantages of PUPTDLSQ we performed numerical calculations of granular distortion for L=1,2,4,8 and rates R=(4-8) bits/sample.
By exceeding L better performances can be achieved but complexity becomes greater.
For L=8 and rates R=(4,6,8),it obtained optimal value of $r_{\max}$, maximal value of $G_{\max}$ and optimal integer values of (L_i,M_i) are given in Table I.
Greater gain can be obtained for greater L and R. For comparing the obtained results to the previous ones, $r_{\max}$ from [5] is used, being obtained for 1-D approach. The corresponding D_g^{scal} is compared to the obtained result using the following gain definition $G = 10\log(D_g^{scal} / D_g^{opt})$.
The performance gain obtained by our method over the uniform scalar quantization for different rates can be presented in this manner: for R=4. G=3.43dB; for R=6, G= 6.05dB and for R=8, G=8.32dB.

TABLE I
OPTIMAL VALUES

L=8	R=4, r_{max}=4.4, Gmax= 3.43dB		R=6, r_{max}=6.6, Gmax= 6.05dB		R=8, r_{max}=8.2, Gmax= 8.32dB	
i	L_i	M_i	L_i	M_i	L_i	M_i
1	5	6	16	20	63	76
2	2	11	11	31	43	123
3	1	9	6	32	26	125
4			3	31	15	110
5			2	20	9	85
6			1	16	5	68
7					3	48
8					2	30

IV CONCLUSION

The optimization of 2-D Laplace source uniform quantization is carried out and the existence of a single minimum depending on the number of points on various levels is proven. Simple expression for granular distortion in closed form is obtained. The results obtained by the asymptotic analysis demonstrate the significant performance gain over the uniform scalar quantization (even 8.32 dB for R=8). The obtained gain using rectangule cells can even be compared to boundary gain in highdimensional space. That automatically provides lower complexity and easier realization.

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