Statistical Parameters of Aperiodic Correlation Functions of Ternary Sequences

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Abstract- **This paper gives an evaluation of the statistical parameters of aperiodic correlation function of ternary sequences. It is shown that the, values distribution of aperiodic correlation functions are hypergeometric, and can be approximate by a normal distribution. Formulas are derived for the mean value and the dispersion of the discussed distributions. The results can be applied to the estimation of error probabilities and capacity of spread spectrum systems, using ternary sequences.**

Keywords- **Ternary sequences, Aperiodic correlation function, Spread spectrum systems.**

I.INTRODUCTION

Main parameters of spread spectrum systems depend on characteristics of pseudorandom sequences or more precisely their correlation functions. This comes from the fact, that they are spreading the spectrum. To minimize the unwanted effects of multipath channel and narrowband jamming, their correlation functions need to be perfect. Perfect crosscorrelation functions define the possibility for code division (CDMA). Despite the periodic correlation functions, the levels of aperiodic correlation functions are also important (partial correlation functions). Decreasing their levels, it will decrease the impact of intersystem interferences in spread spectrum systems and increase their capacity.

Ternary sequences are perspective code class with perfect periodic correlation functions that can be used for spreading the spectrum [1÷6]. This paper defines the basic statistic parameters: mean value and dispersion of ternary sequences aperiodic correlation functions. They have perfect periodic autocorrelation functions and can be generated by polynomial of odd degree n and operation in Galoa field GF(3) [1,2,4,5,6]. The evaluation of statistical parameters will determine the implementation of ternary sequences in spread spectrum systems.

II. BASIC CHARACTERISTICS OF TERNARY SEQUENCES

Let the ternary sequence:

$A=\{\ldots,a_{N-1},a_N,a_1,\ldots,a_{N-1},a_N,\ldots\}$, $a_i\in\{0,-1,+1\}$ за $i = -\infty\ldots+\infty$,

is periodical with a period of N and have an ideal periodic autocorrelation function:

$$R(p)=\frac{1}{N}\sum_{i=1}^{N}a_i a_{i+p}=\begin{cases}\dfrac{P}{N} & p=0(\mathrm{mod}\,N)\\ 0 & p\neq 0(\mathrm{mod}\,N)\end{cases} \quad (1).$$

These sequences can be generated by a polynomial of odd degree n and operation in Galoa field GF(3) [1,2,4,5,6]. Their most important characteristics are:

1. Period $N=(3^n-1)/2$;
2. The periodic autocorrelation function is perfect and maximum of *R(p)* is: $P=3^{n-1}$;
3. Number of zeros in sequence is $k_0=(3^{n-1}-1)/2$, number of +1 is $k_+=(3^{n-1}+3^{(n-1)/2})/2$, and number of –1 is $k_-=(3^{n-1}-3^{(n-1)/2})/2$;
4. $k_+ + k_- = (k_+ - k_-)^2$ - condition for perfect autocorrelation function ;
5. If p is not a divider of the period N, consequently the pair $\{a_i, a_{i+p}\}$ are distributed in the range of N in the following way:
   - a./ $\{1,1\}$ and $\{-1,-1\}$ are met 3^{n-2} times;
   - b./ $\{1,0\}$ and $\{-1,0\}$ are met 2.3^{n-2} times;
   - c./ $\{0,0\}$ is met $(3^{n-2}-1)/2$ times;
   - d./ $\{1,-1\}$ and $\{-1,1\}$ are met 3^{n-2} times;
6. Summering all, the products ($a_i\,.\,a_{i+p}$) taking part in calculation of the periodic autocorrelation function, are distributed as follows:
   - a./ $a_i\,.\,a_{i+p}=0$ is met $Z=(5.3^{n-2}-1)/2$ times;
   - b./ $a_i\,.\,a_{i+p}=+1$ is met $X=3^{n-2}$ times;
   - c./ $a_i\,.\,a_{i+p}=-1$ is met $Y=3^{n-2}$ times;

The aperiodic autocorrelation function is estimated as follows:

$$R_a(p)=\frac{1}{N}\sum_{i=1}^{k}a_i a_{i+p} \quad (2).$$

III.APERIODIC AUTOCORRELATION FUNCTION DISTRIBUTION

Let the calculation of R_a is a random process. Then any product ($a_i\,.\,a_{i+p}$) is random state - $\{a_i, a_{i+p}\}$ are elements of ternary sequence. Because of the periodical autocorrelation function *R(p)*, the set of all random states will consist N elements. Let a random sample has k products ($a_i\,.\,a_{i+p}$) and $k<N$. It consists of x ones (+1), y minus ones (-1) and z zeros (0). The distribution density of x, y and z is a multidimensional hypergeommetric distribution:

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$$f(x,y,z)=\frac{\binom{X}{x}\binom{Y}{y}\binom{Z}{z}}{\binom{N}{k}} \quad (3),$$

$$f(x,y,z)=\frac{\frac{X!}{x!(X-x)!}\ \frac{Y!}{y!(Y-y)!}\ \frac{Z!}{z!(Z-z)!}}{\frac{N!}{k!(N-k)!}} \quad (4).$$

The numbers of +1, -1 and 0 are in the range $0\le x\le X$, $0\le y\le Y$, $0\le z\le Z$. X, Y and Z are defined in characteristic 6.

Let Q is a variable:

$$Q=x-y=N.R_a \quad (5).$$

Using $k=x+y+z$ it is found that:

$$x=\frac{k-z+Q}{2} \quad (6) \qquad \text{and} \qquad y=\frac{k-z-Q}{2} \quad (7).$$

From Eqs. (4), (5), (6) and (7) the two dimensional density of distribution of aperiodic autocorrelation function and the number of zero products in it is:

$$f(Q,z)=\frac{\frac{X!}{\left(\frac{k-z+Q}{2}\right)!\left(X-\frac{k-z+Q}{2}\right)!}\ \frac{Y!}{\left(\frac{k-z-Q}{2}\right)!\left(Y-\frac{k-z-Q}{2}\right)!}\ \frac{Z!}{z!(Z-z)!}}{\frac{N!}{k!(N-k)!}} \quad (8).$$

The case when k is enough big and $Q<<k$ and $z<<k$ is of special interest. For these conditions it is possible to use Starling's formula:

$$\ln k!\approx\left(k+\frac{1}{2}\right)\ln k-k+\frac{1}{2}\ln 2\pi \quad (9),$$

and the decomposition:

$$\ln\left(1\pm\frac{Q}{k-z}\right)\approx\pm\frac{Q}{k-z}-\frac{1}{2}\left(\frac{Q}{k-z}\right)^2 \quad (10).$$

Using these transformations, the distributions (8) tends to a two dimensional normal distribution. This result is obvious taking in to account the central limit theorem, and the fact that for a big sample of k products – x, y and z are random independent.

$$f(Q,z)\approx f_n(Q,z)=\frac{1}{2\pi\sigma_q\sigma_z}\exp\left(-\frac{1}{2}\left(\frac{Q^2}{\sigma_q{}^2}+\frac{(z-m_z)^2}{\sigma_z{}^2}\right)\right) (11).$$

Here $\sigma_q{}^2$ and $\sigma_z{}^2$ are the dispersions of the random quantity. After substitution (8) with (9) and (10) for the dispersion and mean value m_z of z is derived:

$$m_z=k\frac{Z}{N}, \qquad \sigma_z{}^2=4\frac{k(N-k)}{N}\left(\frac{Z}{N}-\frac{X+Y}{N}\right)\frac{Z}{N},$$

$$\sigma_q{}^2=\frac{k(N-k)}{N}\left(1-\frac{Z}{N}\right) \quad (12).$$

The mean value of periodic correlation function m_q is equal to zero. The graph of two dimensional distribution $f(Q,z)$ for $n=7$ and sample of k=40 is shown in fig.1.

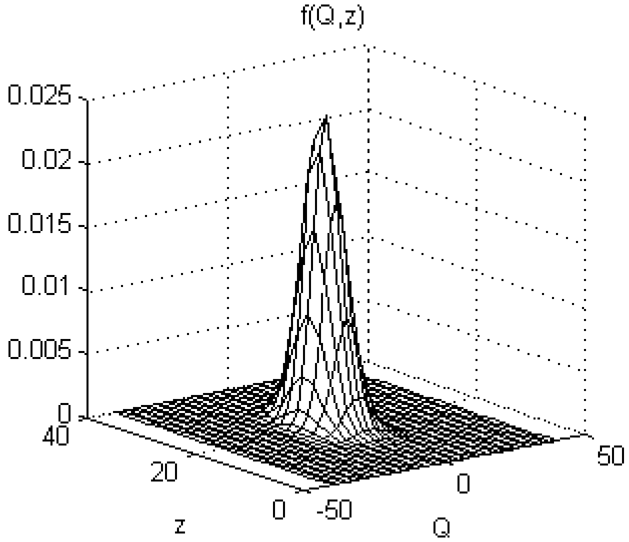

Fig.1

For that the two random variables are independent then the distribution density of Q from (8) is defined as:

$$f(Q)=\sum_{z=1}^{k} f(Q,z) \quad (13).$$

Taking into account (11):

$$f(Q)\approx f_n(Q)=\frac{1}{\sqrt{2\pi\frac{k(N-k)}{N}\left(1-\frac{Z}{N}\right)}}\exp\left(-\frac{Q^2}{2\frac{k(N-k)}{N}\left(1-\frac{Z}{N}\right)}\right) (14).$$

Figure 2 shows the relations $f(Q)/fn(Q)$ for different k. It is obvious the equality of $f(Q)$ and $fn(Q)$ is true for Q<<k, but accuracy is sufficient for the next discussions.

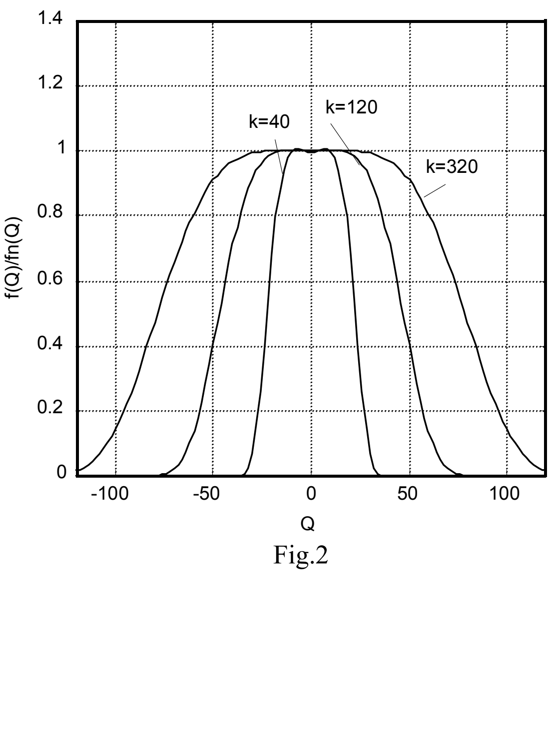

Fig.2

For the density of distribution of the aperiodic autocorrelation function Ra, taking into account (5) and (14) is derived:

$$f_n(R_a) = \frac{N}{\sqrt{2\pi \frac{k(N-k)}{N}\left(1-\frac{Z}{N}\right)}} \exp\left(-\frac{N^2 Ra^2}{2\frac{k(N-k)}{N}\left(1-\frac{Z}{N}\right)} \right) \quad (15),$$

and dispersion :

$$\sigma_{Ra}{}^2 = \frac{k(N-k)(N-Z)}{N^4} \quad (16).$$

The mean value of Ra is equal to zero $m_{Ra}=0$ (17).

IV.RESULTS

The theoretical results can be checked out by the surveying the hystograms of the ternary sequences aperiodic autocorrelation functions. Let calculate:

$$R_a(p) = \frac{1}{N}\sum_{i=1}^{k} a_i a_{i+p} \quad \text{for } p=1 \text{ to } N\text{-}1 \quad (18).$$

Figures 3,4,5 and show the hystogram of ternary sequences aperiodic autocorrelation functions. The sequences are generated by a polynomial of degree n=5 and n=7. They have perfect autocorrelation functions. The same figures show the density of distribution, calculated by (15) too.

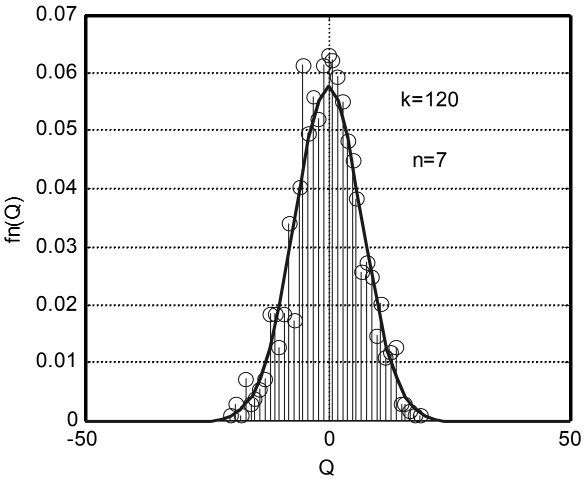

Fig.3

The tend of changing the calculated hystograms is equal to the tend of changing of the theoretically estimated densities of distribution.

Statistic estimation of the mean value is possible by:

$$\hat{m}_{Ra} = \frac{1}{N-1}\sum_{p=1}^{N-1} Ra(p) \quad (19),$$

and dispersion:

$$\hat{\sigma}_q{}^2 = \frac{1}{N-1}\sum_{p=1}^{N-1} (Ra(p) - \hat{m}_q)^2 \quad (20).$$

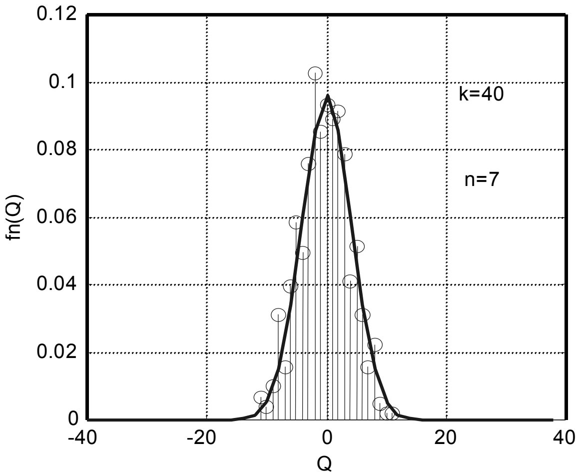

Fig.4

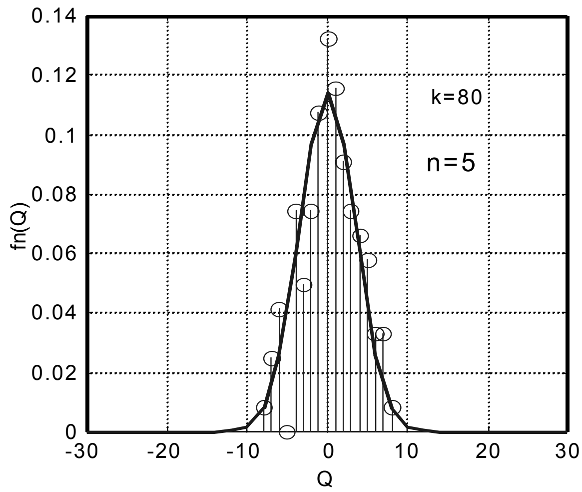

Fig.6

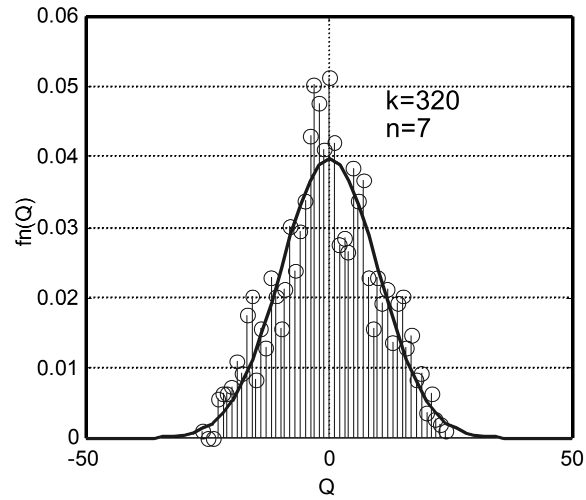

Fig.7

The results sow that mean value is equal to zero as it is by theory in formula (17).

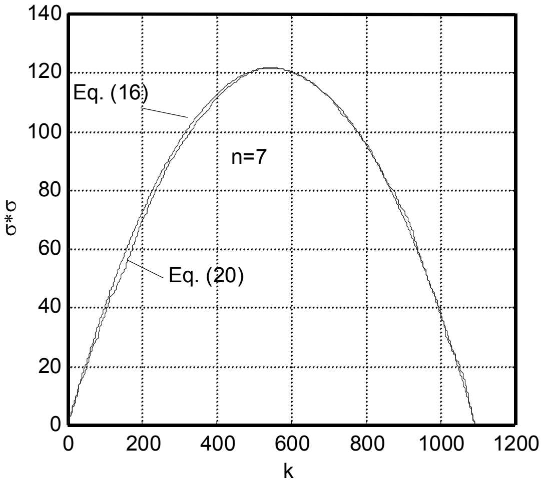

Fig.8

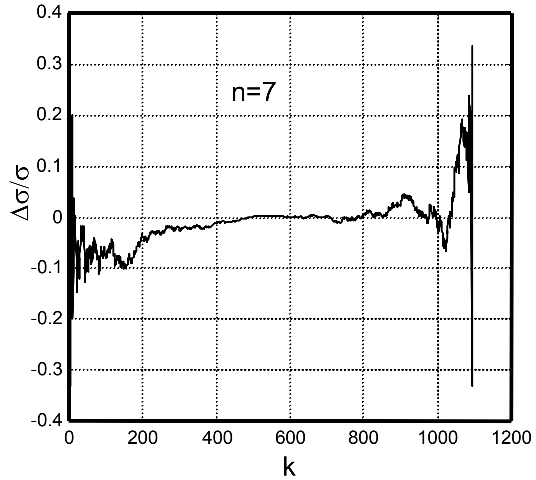

Fig.9

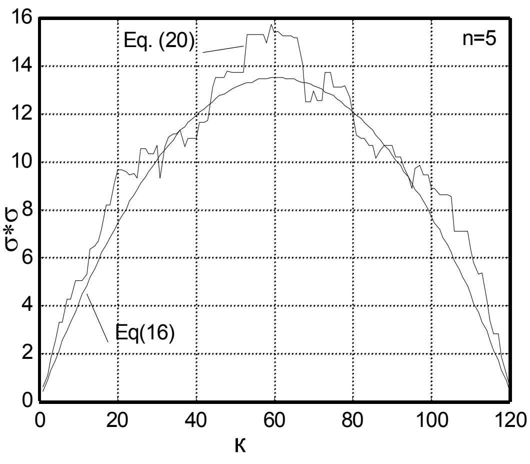

Fig.10

Figures 8 and 9 show the dispersion dependence by sample *k* for *n=7* and *n=5*.

The shown figures are for the theoretical and statistical estimated dispersion by Eq.(16) and Eq(20).

Figures 9 and 11 show the relative of the dispersion from fig.8 and 10.

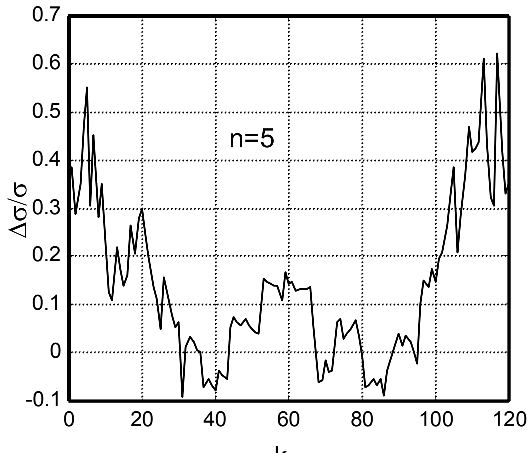

Fig.11

From the results is obvious that, the increasing the period *N* the theoretical dispersion (Eq.16) approaches to the statistical - Eq.20. Theoretical estimated dispersion (Eq.16) can be used as a basic parameter spread spectrum system analysis.

V.CONCLUSION

In this paper is shown, that the distribution of ternary sequences aperiodic correlation function values is hypergeometric and can by approximate by a normal distribution. The ternary sequences have perfect periodic autocorrelation function. It is demonstrated that the mean value of the distribution is equal to zero and the dispersion is a parabolic function of the number of products in aperiodic autocorrelation function and it has maximum by *N/2*.

The derived results can be used in future works to the estimation of error probabilities and capacity of spread spectrum systems, using ternary sequences.

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