A Probabilistic Logic Based on Propositional Intuitionistic Logic

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Abstract - **We introduce a probabilistic extension of propositional intuitionistic logic. The logic allows making statements such as $P_{\geq s}A$, with the intended meaning "the probability of truthfulness of A is greater than or equal to s. We describe the corresponding class of models, which are Kripke models with a naturally arrising notion of probability, and give a sound and complete infinitary axiomatic system. We prove that the logic is decidable.**

Keywords - **probabilistic logic, intuitionistic logic, completeness, decidability**

I. INTRODUCTION

The aim of probabilistic logic is clearly to capture the rules of reasoning about uncertain knowledge. The classical (boolean) logic, on the other hand, may be regarded as starting from a position of omniscience, in so far as it considers that each proposition must be either true or false. Therefore, it is not surprising that combining these two approaches, e.g. by adding probability operators to classical propositional logic as it was done in [4,5,9,10,11], would be sometimes difficult.

There is a popular view of intuitionistic logic as pertaining to growth of knowledge, which is especially convincing when talking about Kripke models. Namely, in addition to propositions which are proved to be true and those which are proved to be false (contradictory), there is a third class of propositions which may turn out either way and intuitionism allows us to reason also about them.

In this paper we combine probabilistic operators with intuitionistic logic. There are two possible approaches to do that. We may treat probabilistic operators classically or we may assume that they behave intuitionistically. The latter approach was analyzed in [2], while we consider here the former one. At the syntax level we add probabilistic operators to the propositional intuitionistic language which enables making formulas such as $P_{\geq s}A$. The intended meaning of the formula is "the probability of truthfulness of A is greater than or equal to s". In our logic nesting of probabilistic operators, i.e., higher order probabilities, will not be allowed.

As a semantics we introduce a class of models that combine properties of intuitionistic Kripke models and probabilities.Since nesting of probabilistic operators is not allowed, it is possible to give a simple and natural interpretation of probabilistic formulas, quite in line with Boole's original ideas, based on the 'size' of the set of possible worlds in which a proposition is true. We propose an infinitary axiomatic system which we prove is sound and complete with respect to the mentioned class of probabilistic intuitionistic models.

In this paper the terms *finitary* and *infinitary* concern meta language only. Object languages are countable, formulas are finite, while only proofs are allowed to be infinite. The need for this infiniteness comes from the failure of compactness theorem for this type of logics, as will be explained in the conclusion.

We may try to motivate this combination of intuitionistic and probabilistic logics through the following example. It is well known that $(p \rightarrow q) \vee (q \rightarrow p)$ is a classical, but not intuitionistic, tautology. Since tautologies should have probability one, starting with classical logic makes $P_{\geq 1}((p \rightarrow q) \vee (q \rightarrow p))$ valid in probabilistic logic. If we take now p to be "it rains" and q to be "the sprinkler is on", it is clear that the sprinkler should not be on when it rains, i.e., $p \rightarrow q$ should low probability, say less than E. But, this yields, with classical logic, $P_{\geq 1-E}(q \rightarrow p)$ (since the measure of the union of two sets is less or equal than the sum of the measures of those sets) although it does not seem highly probable that it should rain when the sprinkler is on.

For those without experience with intuitionistic logic, one should stress that $p \rightarrow q$ being false in a Kripke model does not mean that it is always rainig but only that in every world there is some later world (world with more information) in wich it does rain but the sprinkler is off. Therefore, we may also have quite a few worlds in which the sprinkler is on but it does not rain.

Comming back to the argument we made at the begining, from the point of view of classical logic it either rains or it does not. The intuitionistic logic allows us to be in a world in which we simply do not know which is true but from which we can envisage some possible worlds in which it does rain (and maybe also some in which it does not). Simply counting the worlds in which $p \rightarrow q$ is true (and dividing by the total number of worlds) may give us the probability of its truthfullness in a given model. In such a way we may construct models in which both $p \rightarrow q$ and $q \rightarrow p$ have low probability which is impossible if we work with classical logic.

As for the classical treatment of probabilistic operators, we may say that once we determine (somehow) the probability of an uncertain proposition A, it should be either greater or equal

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to some s from [0,1] or not, so it is not unreasonable to assume $P_{\geq s}A \vee \neg P_{\geq s} A$ (even if we reject A V ¬A).

II. Syntax

Let S be the set of all rational numbers from [0, 1]. The language of the logic consists of a denumerable set F = {p, q, r, ...} of propositional letters, connectives ¬, &, V, → and a list of unary probabilistic operators $(P_{\geq s})_{s\in S}$. The set For(I) of intuitionistic propositional formulas is the smallest set X containing F and closed under the formation rules: if A and B belong to X, then ¬A, A & B, A V B, and A → B are in X. The set For(P) of probabilistic propositional formulas is the smallest set Y containing all formulas of the form $P_{\geq s}A$ for A ∈ For(I), s ∈ S, and closed under the formation rules: if A and B belong to Y, then ¬A, and A & B are in Y. Let For(I) U For(P) be denoted by For. Probabilistic literals are formulas of the form $P_{\geq s}A$ or $\neg P_{\geq s}A$. We use A V B, A → B and $P_{<s}$ A to denote the formulas ¬(¬A & ¬B), ¬A V B and $\neg P_{\geq s}A$, respectively. Note a difference between the sets For(I) and For(P). Namely, according to the previous definitions, propositional connectives are independently introduced in the set For(I), while V and → are defined from ¬ and & in the set For(P).

III. Semantics

We propose a possible-world approach to give semantics to formulas from the set For. According to the structure of For, there are two levels in the definition of models. At the first level there is the notion of intuitionistic Kripke models [8], while probability comes in the picture at the second level.

Definition 1. An intuitionistic (propositional) Kripke model for the language For(I) is a structure < W, ⊆, v > where:

- < W, ⊆ > is a partially ordered set of possible words which is a tree, and
- v is a valuation function, i.e., v maps the set W into the powerset P(F)), which satisfies the condition: for all w, w' from W, w ⊆ w' implies v(w) ⊆ v(w').

We find it convinient to work with Kripke models in which the ordering of worlds is a tree. It is well known that intuitionistic propositional logic is sound and complete also with respect to this restricted class of Kripke models.

In each Kripke model we define the forcing relation ||- by the following definition:

Definition 2. Let < W, ≤, v > be an intuitionistic Kripke model. The forcing relation ||- is defined by the following conditions for every w from W, A, B from For(I):

- for A from F, w ||- A iff A ∈ v(w),
- w ||- A & B iff w ||- A and w ||- B,
- w ||- A V B iff w ||- A or w ||- B,
- w ||- A → B iff for every w' from W if w ⊆ w' then it is not w' ||- A or w' ||- B, and
- w ||- ¬ A iff for every w' from W if w ⊆ w' then it is not w'||- A.

We read w ||- A as "w forces A" or "A is true in the world w". Validity in the intuitionistic Kripke model < W, ⊆, v> is defined by < W, ≤, v> |= A iff for all w from W, w ||- A. A formula A is valid (|= A) if it is valid in every intuitionistic Kripke model.

Let M = < W, (, v> be an intuitionistic Kripke model. We use [A]M to denote the set of all w from W such that w ||- A, for every A from For(I). Clearly, the family HI= {[A]M: A (For(I)} is a Heyting algebra. Thus, H_I is a lattice on W, but it may not be closed under complementation.

Definition 3. A probabilistic model is a structure < W, (, v, H, m > where:

- < W, ≤, v > is an intuitionistic Kripke model,
- H is the smalest algebra on W containing HI
- m : H → [0, 1] is a finitely additive probability.

Note that H contains all sets of the form W \ [A]M, even if for some A from For(I) it may not be W \ [A]M = [¬A]M.

Definition 4. The satisfiability relation |= is defined by the following conditions for every probabilistic model M = <W,(,,v,H,m >:

- for A from For(I), M |=A if for all w from W, w ||- A,
- M |= P≥sA if m([A]M) ≥ s,
- for A from For(P), M|=¬A if M|=A does not hold,
- for all A, B from For(P), M |= A & B if M |= A, and M |= B.

A formula A from For is satisfiable if there is a probabilistic model M such that M |= A; A is valid if for every probabilistic model M, M |= A; a set of formulas is satisfiable if there is a probabilistic model M such that for every formula A from the set, M |= A.

IV. A sound and complete axiomatization

The set of all valid formulas can be characterized by the following sound and complete set of axiom schemata:

1. all For(I)-instances of intuitionistic propositional tautologies
2. all For(P)-instances of classical tautologies
3. P≥0 A
4. P≥1-r ¬A → ¬P≥s A, for s > r
5. P≥r A → P≥s A, for r ≥ s
6. P≥1(A → B) →(P≥sA → P≥sB)
7. (P≥s A & P≥r B & P≥1¬(A&B)) → P≥ min(1,s+r)(A V B)
8. (P< s A & P<r B → P<s+r(A V B), s+r ≤ 1
9. and inference rules:
10. Modus ponens
11. If A (For(I), from A infer P≥1 A.
12. From B→P≥s-1/k A, for every k≥ 1/s, infer B → P≥s A.

A formula A is deducible from a set T of formulas (T |- A) if there is an at most countable sequence of formulas A_0, A_1,

..., A, such that every formula in the sequence is an axiom or a formula from the set T, or it is derived from the preceding formulas by an application of an inference rule. If T |- A and T is an emptyset, we say that A is a theorem of the deductive system (denoted by |- A).A set T of formulas is consistent if it is not T |- ¬ (A → A). Otherwise, T is inconsistent.

A set T of formulas is deductively closed if for every A from For, if T |- A, then A belongs to T. A set T of formulas has the disjunction property if for every A, B from For(I), T |- A V B implies T |- A or T |- B. A disjunctive closure of a set T of formulas is a set T' which contains T and for every A, B from For(I), if A V B belongs to T then A belongs to T' or B belongs to T'.

Soundness theorem 1. The above axiomatic system is sound with respect to the class of probabilistic models.

Proof. Soundness of our system follows from the soundness of propositional intuitionistic and classical logics, as well as from the properties of probabilistic measures.

In the proof of the completeness theorem the following strategy is applied. We start with a form of Deduction theorem. Then, we show how to extend a consistent set T of formulas to a consistent set T^* which is in some sense maximal. Then, a canonical model M is constructed out of the formulas from the set T^* such that M |= A iff A belongs to T^*.

Deduction theorem 2. If T is a set of formulas and T U {A} |- B, then T |- A → B, where either A, B belong to For(I) or A, B belong to For(P).

We now describe how to extend a consistent set T of formulas in a proper way. Let ipconseq(T) = { A ∈ For(I): T |- A } be the set of all intuitionistic propositional consequences of T, and T'be a consistent disjunctive closure of ipconseq(T). Let A_0, A_1, … be an enumeration of all formulas from For(P).We define a sequence of sets T_i, i = 0, 1, 2, ..., and a set T^* such that:

- T_0= T U T' U {$P_{\geq 1}$A: A ∈ T'}
- for every i = 0, 1, …, if T_i U {A_i} is consistent, then T_{i+1} = T_i U {A_i}, otherwise, T_{i+1} = T_i U
- if T_{i+1} is obtained by adding a formula of the form ¬(B → $P_{\geq s}$C), then for some positive integer n, B → ¬$P_{\geq s-1/n}$C, is also added to T_{i+1}, so that T_{i+1} is consistent,
- T^* = U_i T_i.

The set T^* is used to construct a canonical probabilistic model M. First, we construct an intuitionistic Kripke model. Let w_0 denote the set T' and W be the set of all consistent, deductively closed extensions of w_0 having the disjunction property. Let v(w) = {A ∈ F: A ∈ w}. Then < W, ⊆, v > is an intuitionistic Kripke model, and for every w ∈ W, and every A ∈ For(I), w ||- A iff A ∈ w.

In the next step we define a canonical probabilistic model M.Let < W, ⊆, v > be as above, H_I = $\{[A]_M\}_{A \in For(I)}$, and for every A ∈ For(I), $m_I([A]_M)$ = sup{s: $P_{\geq s}$ A ∈ T^*}. Let H be the smalest algebra on W containing H_I, and m a finitely additive probability on H which is an extension of m_I.

Theorem 3. M = < W, ⊆, v, H, m > is a probabilistic model.

Completeness theorem 4. Every consistent set of formulas is satisfiable.

V. DECIDABILITY

It is well known that a formula A ∈ For(I) is intuitionistically satisfiable iff it is forced in the root of a tree-like model which is decidable [6,8,12]. It follows that satisfiability problem of For(I)-formulas in our probabilistic logic is decidable. Thus, to prove decidability of our logic it is enough to show that satisfiability problem for probabilistic formulas is decidable.

Let A ∈ For(P) and Sub(I)(A) = { A ∈ For(I): A is a subformula of A}. Let |A| and |Sub(I)(A)| denote the length of A, and the number of formulas in |Sub(I)(A)|, respectively. Obviously, |Sub(I)(A)| ≤ |A|.

Theorem 5. A probabilistic formula A (For(P) is satisfiable iff it is satisfiable in a finite probabilistic model containing at most 2(|A|^2) worlds.

Theorem 6. The satisfiability problem for probabilistic formulas is decidable.

Proof. For A (For(P) let us denote by DNF(A) the formula Vi (&j #P(s(i,j) Ai,j), where #P(s(i,j) Ai,j's are probabilistic literals, which is equivalent to A. For every A (For(P) there is at least one DNF(A), because propositional connectives behave classically at the probabilistic level. A is satisfiable iff at least one disjunct D from DNF(A) is satisfiable. Since D is a conjunction of probabilistic literals, without loss of generality we can assume that A is of the same form. We know that A is satisfiable iff it is satisfiable in a probabilistic model with at most k(A) = 2(|A|^2) worlds. Thus, we can check satisfiability of A in the following way.For every integer l from [1, k(A)], there is only finitely many intuitionistic models with different valuations with respect to the set of propositional letters that occur in A. For every such intuitionistic model M(I) = < W, (, v > we can find the algebra H generated by the set {[A]M(I): A (Sub(I)(A)} and suppose that every {w} (w (W) belongs to H as well, and consider the following linear system:

∑w (W m(w) = 1

m(w) (0, for w (W

∑w ([A]_M(I) m(w) (r, for every P(rA which appears in A

$\sum_{w \in [A]\_M(I)}$ m(w) < r, for every ¬$P_{\geq r}$ A which appears in A.

Obviously, if the above system is solvable, M = < W, ⊆, v, H, m > |= A. There is a finite number of models and linear systems we have to check. Since linear programming problem is decidable, the same holds for the considered satisfiability problem. Finally, since A is valid iff ¬A is not satisfiable, the validity problem is also decidable.

VI. CONCLUSION

In this paper we have investigated a logic which combines probabilistic and intuitionistic reasoning. We have given an axiomatic system which is sound and complete with respect to a class of Kripke-style models. We have also proved that our logic is decidable. The compactness theorem does not hold for our logic. To see that consider the set T = { $\neg P_{\geq 1} A$ } U { $P_{\geq 1 - 1/n}$ A: n is a positive integer }. Although every finite subset of T is satisfiable, the set T itself is not. Since the compactness theorem follows easily from the extended completeness theorem ('every consistent set of formulas is satisfiable'), we cannot hope for the extended completeness when we have a finitary axiomatic system. However, including an infinitary rule in our axiomatization we obtain the extended completeness theorem. As we have already noted this is not the first paper which considers combinations of intuitionistic and probabilistic logics. In [2,3] two kinds of probabilistic operators for upper and lower probabilities were added to Heyting propositional logic. The corresponding models were Kripke models with two families of measures that were subadditive and superadditive, respectively, and monotone with respect to the order of the worlds. An axiomatic system was presented and showed to be sound and complete for the models described. In that logic iteration of probabilistic operators was allowed. On the other hand, our probabilistic operators behave classically which enables that in every model there is only one probability distribution. Since we allow probabilistic operators to be applied to propositional formulas from For(I) only, it would be interesting to see whether our approach can be used when higher order probabilities are considered in intuitionistic framework. Our approach is more similar to papers [4,5,9,10,11]. However, since the basic logic in those papers is classical propositional logic, and we start from intuitionistic logic, there are formulas that are valid in the mentioned logics, but not in ours. In [9,10,11] probabilistic operators of the forms $P_{\leq s}$ and $P_{> s}$ (meaning "probability is at most s", and "probability is greater than s") were defined as $P_{\leq s} A = P_{\geq 1-s} \neg A$ and $P_{>s} A = \neg P_{\geq 1-s} \neg A$, respectively. It can not be done in the approach presented here. Namely, $P_{\leq s} \neg A$ follows from P(1-s A, but it is easy to see that the other direction does not hold. Instead, we may extend our logic, in the spirit of intuitionistic systems, by having two kinds of probabilistic operators of the forms $P \geq s$, and $P \leq s$, that are not inter-definable, and by adding corresponding duals of the axioms and inference rules.

Finally, we note that working with real numbers, supremum etc., as we have done above, may be seen as suspect from the intuitionistic point of view. To avoid such a problem, we can apply the following idea and sintactically control ranges of probabilities. Let S be a recursive subset of [0, 1] which contais all rational numbers from that interval. We extend the propositional language with a list of probabilistic operators (P≥s)s(S. Next, we demand that ranges of probabilities in probabilistic models must be subsets of S. Otherwise, all the definitions about models and satisfiability remain the same as above. To obtain a complete axiomatization with respect to the new class of models a new inference rule:

- From A → P≠s A, for every s (S, infer A → false.

should be substituted for the only infinitary rule in the axiomatic system.

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