Some Remarks on Complete Theory of One Probability Logic

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Abstract - **We give the representation for one class of probability algebras whose set algebras are constructed with respect to the probability measure spaces. On the other hand a formula algebra of one probability logic belongs to the same class such that it is possible to define the probabilities of formulas as measures of corresponding sets.**

Key words: **weak probability cylindric algebra, probability logic, probability of formulas.**

I. INTRODUCTION

Since there is no possibility of confusion, the concatenation of two sequences $\bar{x}$ and $\bar{y}$ where $\bar{y} = \langle a \rangle$ is 1-termed and $a \notin Rg\bar{x}$ will be denoted with $\bar{x}a$. The set of one-one function from $n \in \omega$ into an ordinal α is denoted with $\alpha^n_{\mapsto}$, and $\alpha^{<\omega}_{\mapsto}$ denotes $\bigcup_{n\in\omega} \alpha^{<\omega}_{\mapsto}$.

Weak probability cylindric algebra of dimension α, where α is any ordinal, is cylindric algebra of dimension α enriched by unary operations C^r_ρ for $r \in [0,1]$ and $\rho \in \alpha^{<\omega}_{\mapsto}$ which are called probability cylindrifications. The class of all loccally finite weak probability cylindric algebras of dimension α is denoted with $\mathbf{wpLf}_\alpha$ and defined in exactly the same way as in the theory of cylindric algebras.

DEFINITION: Let U be a set and α any ordinal. Suppose $X \subseteq U^\alpha$, $u \in U^\alpha$ and $\rho \in \alpha^{<\omega}_{\mapsto}$. The set

$$X^{[u]_\rho} = \left\{ x \circ \rho : x \in X, x|\backslash \alpha \sim Rg\rho = u|\backslash \alpha \sim Rg\rho \right\}$$

is said to be $u - \rho$ cross section of X.

A weak probability cylindric set algebra of dimension α (with respect to the finitely additive n-fold product probability measure spaces for $n \in \omega \sim 1$) is a structure

$$\mathsf{A} = \left\langle \boldsymbol{A}, \cup, \cap, \sim, 0, U^\alpha, \boldsymbol{C}_\kappa, \boldsymbol{C}^r_\rho, \boldsymbol{D}_{\kappa\lambda} \right\rangle$$

such that $\kappa, \lambda < \alpha$, $\rho \in \alpha^{<\omega}_{\mapsto}$, $r \in [0,1]$ and $\boldsymbol{A}$ is a non-empty subset of SbU^α, and such that the following holds:

(1) There is a finitely additive probability measure space $\mathsf{U} = \langle U, S, \mu \rangle$ such that $\left\langle U^n, S^{(n)}, \mu_n \right\rangle$ is a finitely additive n-fold product probability measure space of U and for each $n \in \omega \sim 1$,

$$\left\{ X^{[u]_\rho} : X \in \boldsymbol{A}, \rho \in \alpha^n_{\mapsto}, u \in U^\alpha \right\} \subseteq S^{(n)}. \qquad (2)$$

$\boldsymbol{C}^r_\rho : SbU^\alpha \to SbU^\alpha$ for any $\rho \in \alpha^{<\omega}_{\mapsto}$ and any $r \in [0,1]$ such that for every $X \subseteq U^\alpha$ we have $\boldsymbol{C}^r_0 X = X$, and if $Do\rho = n$, then

$$\boldsymbol{C}^r_\rho X = \left\{ u \in U^\alpha : \mu_n \left(X^{[u]_\rho} \right) \geq r \right\}. \qquad (3)$$

The collection $\boldsymbol{A}$ is an $\alpha -$dimensional cylindric field of sets and $\boldsymbol{A}$ is closed under probability operations $\boldsymbol{C}^r_\rho$ (for all $r \in [0,1]$ and all $\rho \in \alpha^{<\omega}_{\mapsto}$).

II REPRESENTATION THEOREM

Some result will be mentioned which have the role in establishing sufficient conditions for representability of simple rich algebras from the class $\mathbf{wpLf}_\alpha$.

Suppose $\mathsf{A} \in \mathbf{wpLf}_\alpha$, $x \in A$ and $\rho \in \alpha^{<\omega}_{\mapsto}$. By the characteristic point of x with respect to ρ, in symbol $ch_\rho(x)$, is meant $\max\left\{ r : r \in [0,1], C^r_\rho x = 1 \right\}$. This max always exists. It is not hard to show that A is simple iff contains just two zero-dimensional elements. This enables us to prove that if A is simple and $\Delta x \subseteq Rg\rho$ (Δx is the set of all ordinals $\mu < \alpha$ for which $c_\mu x \neq x$), then for each $r \in [0,1]$ either $C^r_\rho x = 0$ or $C^r_\rho x = 1$.

If A is assumed to be simple, we then have:

1. $ch_\rho(x) = \inf\left\{ r : r \in [0,1], C^r_\rho x = 0 \right\}$, if $x \neq 1$.
2. If $\rho \neq 0$ and ch_ρ is understood as a function from the set of all elements of A for which $\Delta x \subseteq Rg\rho$ into $[0,1]$, then ch_ρ is a finitely additive function.
3. If $P = \left\langle x_j : j \in J \right\rangle$ is an arbitrary system of pairwise disjoint elements of A such that $\Delta x_j \subseteq Rg\rho$ for

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each $j \in J$ and Q is the set of all members x of P for which $ch_\rho(x) > 0$, then Q is at most countable and

$$\sum_{x \in Q} ch_\rho(x) \le 1.$$

4. If $\alpha \ge \omega$ and A is a set algebra, then elements of $\boldsymbol{A}$ are similar to the tail sets and for any $\rho \in \alpha^n_{\mapsto}$ and any $X \in \boldsymbol{A}$ such that $\Delta X \subseteq Rg\rho$ we have that

$$\mu_n\left(\left\{ x \middle\backslash Rg\rho : x \in X \right\}\right) = ch_\rho(X).$$

It should be mentioned that the third statement is extended to arbitrary probability algebras.
The notions of the rich algebra and the 0-thin element are taken over the theory of cylindric algebras.
THEOREM: Suppose $\mathsf{A} \in \mathbf{wpLf}_\alpha$ with $\alpha \ge \omega$ and A is simple rich having the rectangle property. Let U be the set of all 0-thin elements in A and

$$\sum_{x \in Q} ch_{\langle 0 \rangle}(x) = 1, \text{ where}$$

$$Q = \left\{ u : u \in U, ch_{\langle 0 \rangle}(u) > 0 \right\}.$$

Then A is isomorphic to a weak probability cylindric set algebra such that μ_n's are countably additive.
It should be pointed out that the proof which we given is continued the proof of analogous theorem for cylindric algebras (see [1]). Our proof extends an algebraic version of Henkin's proof of Completeness Theorem for first-order predicate calculus to the probability case.

III. PROBABILITY LOGIC L_{WOP}

The logic we want to define will be called the weak ordinary probability logic and denoted with L_{WOP}.
A language L of this logic contains only finitary relations and constant symbols. The sequence of variables $\langle x_\xi : \xi < \alpha \rangle$ has infinite length.
The connectives $\neg$ and $\bigwedge$, the ordinary quantifiers for all variables, the probability quantifiers $(P\overline{x}_\rho \ge r)$ for all $r \in [0,1]$ and all $\rho \in \alpha^{<\omega}_{\mapsto}$ such that

$$\overline{x}_\rho = \left\langle x_{\rho(0)}, \cdots, x_{\rho(n-1)} \right\rangle,$$

where $Do\rho = n$, and the equality symbol $=$ are logical symbols of L_{WOP}. Also, the truth symbol T and the falsehood symbol F are logical symbols treated as sentential constants. The finitary connectives $\vee$, $\wedge$, $\rightarrow$ and $\leftrightarrow$ are defined as usual.
$\bigvee_{\varphi \in \Phi} \varphi$ is an abbreviation for $\neg \bigwedge_{\varphi \in \Phi} \neg\varphi$.

The set $Form^L$ of formulas of L_{WOP} in language L is the least set such that: each atomic formula of ordinary first-order logic in L is a formula of L_{WOP}, if φ is a formula of L_{WOP}, then $\neg\varphi$, $\forall x\varphi$, $\exists x\varphi$ (for all variables x) and $(P\overline{x} \ge r)\varphi$ for all probability quantifiers are formulas of L_{WOP}, and if Φ is a countable set of formulas of L_{WOP} with only finitely many variables, then $\bigwedge \Phi$ is a formula of L_{WOP}.
The set of axioms for L_{WOP} contains all formulas of the following forms:
(A1) All axioms of ordinary first-order logic (in L);
(A2) $\wedge \Phi \rightarrow \varphi$, if $\varphi \in \Phi$;
(A3) $(P\overline{x} \ge r)\varphi \rightarrow (P\overline{x} \ge s)\varphi$, if $r \ge s$;
(A4) $(P\overline{x} \ge r)\varphi(\overline{x}) \rightarrow (P\overline{y} \ge r)\varphi(\overline{y})$.
(A5) $(P\overline{x} \ge 0)\varphi$.
(A6) $(P\overline{x} \ge r)\varphi \wedge (P\overline{x} \ge s)\psi \rightarrow$
$\rightarrow (P\overline{x} \ge \max(0, r+s-1)(\varphi \wedge \psi)$.
(A7) $(P\overline{x} \ge r)\varphi \wedge (P\overline{x} \ge s)\psi \wedge (P\overline{x} \ge 1)$
$(\neg(\varphi \wedge \psi)) \rightarrow (P\overline{x} \ge \min(1, r+s))(\varphi \vee \psi)$.
(A8) $\neg(P\overline{x} \ge r)\neg\varphi \leftrightarrow \bigvee_{n \in \omega \sim 1} \left(P\overline{x} \ge 1 - r + \frac{1}{n} \right)\varphi$, if $r \ne 0$.
(A9) $(P\overline{x} \ge r)\varphi \leftrightarrow (P\overline{y} \ge r)\varphi$, if $Rg\overline{x} = Rg\overline{y}$.
(A10) $\neg(P\langle x_K \rangle \ge r)\neg\varphi \rightarrow \exists x_K \varphi$.
(A11) Provided $Rg\overline{x} \cap Rg\overline{y} = 0$ and the set of all free variables of φ, respectively ψ is a subset of $Rg\overline{x}$, respectively $Rg\overline{y}$,
$(P\overline{x} \ge r)(P\overline{y} \ge s)(\varphi \wedge \psi) \rightarrow (P\overline{xy} \ge r \cdot s)(\varphi \wedge \psi)$.
The Rules of Inference are as follows: Modus Ponens, Universal Generalization, Conjuction and Probability Generalization.
The Deduction Theorem holds. Concerning the Completeness Theorem the answer is positive for a similar logic $L_{\mathbf{A}P\forall}$, where $\mathbf{A}$ is a countable admissible set, $\omega \in \mathbf{A}$, the length of the sequence of variables is ω and the formulas are constructed set theoretically each of which belongs to $\mathbf{A}$.
The theorems of L_{WOP} concerning only probability part are analogous to the theorems of $L_{\mathbf{A}P}$ (see [2]). The list below does not contain them except of first three which will be involved in an example of provability for one formula.
Thus the following are theorems of L_{WOP}.
(1) If $\vdash \varphi \rightarrow \psi$, then $(P\overline{x} \ge r)\varphi \rightarrow (P\overline{x} \ge r)\psi$.
(2) If variables from $\overline{x}$ does not free occur in φ, then
$\vdash (P\overline{x} \ge r)(\varphi \wedge \psi) \leftrightarrow \varphi \wedge (P\overline{x} \ge r)\psi$ and
$\vdash (P\overline{x} \ge r)(\varphi \vee \psi) \leftrightarrow \varphi \vee (P\overline{x} \ge r)\psi$.
(3) If $x_K \notin Rg\overline{x}$, then
$\vdash (P\overline{x}x_K \ge r)\varphi \rightarrow (P\overline{x} \ge r)\exists x_K \varphi$.
(4) If $x_K \notin Rg\overline{x}$, then
$\vdash (P\overline{x} \ge r)\exists x_K \varphi \leftrightarrow (P\overline{x}x_K \ge r)\exists x_K \varphi$.
(5) $\vdash \neg(P\overline{x} \ge 1)\neg\varphi \rightarrow \exists\overline{x}\varphi$.

(6) $\vdash (P\bar{x} \geq r)\varphi \rightarrow \exists\bar{x}\varphi.$

(7) $\vdash \forall\bar{x}\varphi \rightarrow (P\bar{x} \geq r)\varphi.$

(8) $\vdash (P\bar{x} \geq r)\forall x_\kappa \varphi \rightarrow \forall x_\kappa (P\bar{x} \geq r)\varphi.$

(9) $\vdash \exists x_\kappa (P\bar{x} \geq r)\varphi \rightarrow (P\bar{x} \geq r)\exists x_\kappa \varphi.$

(10) If $x_\kappa \in Rg\bar{x}$, then
$\vdash \exists x_\kappa (P\bar{x} \geq r)\varphi \leftrightarrow (P\bar{x} \geq r)\varphi.$

(11) If $x_\kappa, x_\lambda \notin Rg\bar{x}$, then
$\vdash (P\bar{x} \geq r)\exists x_\kappa (x_\kappa = x_\lambda \wedge \varphi) \leftrightarrow$
$\leftrightarrow \exists x_\kappa (P\bar{x} \geq r)(x_\kappa = x_\lambda \wedge \varphi).$

(12) If $x_\kappa, x_\lambda \notin Rg\bar{x}$, then
$\vdash (P\bar{x}x_\kappa \geq r)\exists x_\lambda (x_\kappa = x_\lambda \wedge \varphi) \leftrightarrow$
$\leftrightarrow (P\bar{x}x_\lambda \geq r)\exists x_\kappa (x_\kappa = x_\lambda \wedge \varphi).$

As an example we sketch the proof of (11) using well known result for first-order logic, that is,

(13) $\vdash \exists x_\kappa (x_\kappa = x_\lambda \wedge \varphi) \leftrightarrow \forall x_\kappa (x_\kappa \neq x_\lambda \vee \varphi).$

Denote $(P\bar{x} \geq r)\exists x_\kappa (x_\kappa = x_\lambda \wedge \varphi)$ with θ.

Making use of (1) and (13) we get

$\vdash \theta \rightarrow (P\bar{x} \geq r)\forall x_\kappa (x_\kappa \neq x_\lambda \vee \varphi).$

Using (8) and after this the second part of (2) it follows

$\vdash \theta \rightarrow \forall x_\kappa (P\bar{x} \geq r)(x_\kappa \neq x_\lambda \vee \varphi)$

$\vdash \theta \rightarrow \forall x_\kappa (x_\kappa \neq x_\lambda \vee (P\bar{x} \geq r)\varphi).$

Applying (13) we show

$\vdash \theta \rightarrow \exists x_\kappa (x_\kappa = x_\lambda \wedge (P\bar{x} \geq r)\varphi).$

Now, one use of the first part of (2) gives the desired implication. The opposite is an immediate from (9).

IV. METALOGICAL CONSEQUENCES OF MAIN THEOREM

We look at the relationship between the theorem in Section 2. and probability logic L_{WOP}.

Let Σ be a complete consistent theory of logic L_{WOP} (in some language *L*). Let $Fm^L_{\equiv\Sigma}$ be the algebra of formulas associated with Σ, i.e.,

$$Fm^L_{\equiv\Sigma} = \left\langle \mathrm{Form}^L_{/\equiv\Sigma}, \vee^\Sigma, \wedge^\Sigma, \neg^\Sigma, F^\Sigma, T^\Sigma, (x_\kappa = x_\lambda)^\Sigma, (P\bar{x}_\rho \geq r)^\Sigma \right\rangle_{\kappa,\lambda<\alpha,\ \rho\in\alpha^{<\omega}_{\mapsto},\ r\in[0,1]}.$$

Having the above mentioned theorems it is not difficult to verify that $Form^L_{\equiv\Sigma} \in \mathbf{wpLf}_\alpha$. The completeness of Σ together with the remark at the beginning of the Section 2. asserts that $Form^L_{\equiv\Sigma}$ is simple. The axiom (A11) implies that the same algebra has the rectangle property.

Denote the set of individual constants with C. As in a first-order logic, the equivalence classes $(x_0 = c)_{\equiv\Sigma}$ for $c \in C$ are 0-thin elements.

Let the next condition be satisfied for each formula $\varphi(x_0) \in Form^L$: if $\Sigma \vdash \exists x_0 \varphi(x_0)$, then there exists $c \in C$ such that $\Sigma \vdash x_0 = c \wedge \exists x_0 \varphi(x_0) \rightarrow \varphi(x_0)$.

This condition means that our formula algebra is rich.

It can be proved that if $\varphi \in Form^L$, then

(1) $$ch_\rho(\varphi_{\equiv\Sigma}) = \max\left\{ r : \Sigma \vdash (P\bar{x}_\rho \geq r)\varphi \right\}.$$

We set

$$Q = \left\{ c : c \in C, ch_{\langle 0\rangle}\left((x_0 = c)_{\equiv\Sigma}\right) > 0 \right\}.$$

Suppose $\sum_{c\in Q} ch_{\langle 0\rangle}\left((x_0 = c)_{\equiv\Sigma}\right) = 1.$

Then, on the basis of the theorem in Section 2. there exists an isomorphism h from $Form^L_{\equiv\Sigma}$ onto a weak probability cylindric set algebra of dimension α.

For each $\rho \in \alpha^{<\omega}_{\mapsto}$ we let $Form^L_\rho$ be the set of all formulas $\varphi(\bar{x}) \in Form^L$ such that

$$\varphi(\bar{x}) \in Form^L_\rho \text{ iff } Rg\bar{x} \subseteq Rg\bar{x}_\rho.$$

Using the fact that $ch_\rho\left(h(\varphi_{\equiv\Sigma})\right) = ch_\rho(\varphi_{\equiv\Sigma})$ for each $\varphi \in Form^L$ and the result 2.4. it is possible that the set $\left\{\varphi_{\equiv\Sigma} : \varphi \in Form^L_\rho\right\} \subseteq \mathrm{Form}^L_{/\equiv\Sigma}$ carry over the probability laws of certain product probability space.

According to (1), for each $\rho \in \alpha^{<\omega}_{\mapsto}$ it could be defined the probability on the set $Form^L_\rho$.

It is natural to suggest that the investigation on the decidability of complete theories of this probability logic could be useful not only for working with probability of formulas related to a complete theory Σ, but for measure problems solving, too.

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