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Plenary Lecture

Third-order Tensor Decomposition through Inverse Spectrum Pyramid: Algorithms and Applications

Roumen Kountchev, Prof., D.Sc., PhD

Roumiana Kountcheva, PhD

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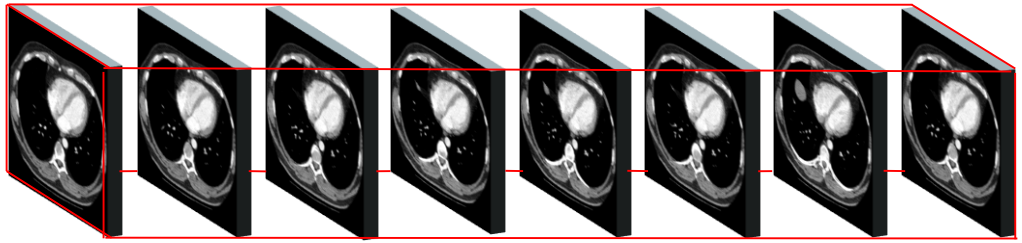
Outlines

1. Introduction: 3D tensor image representation

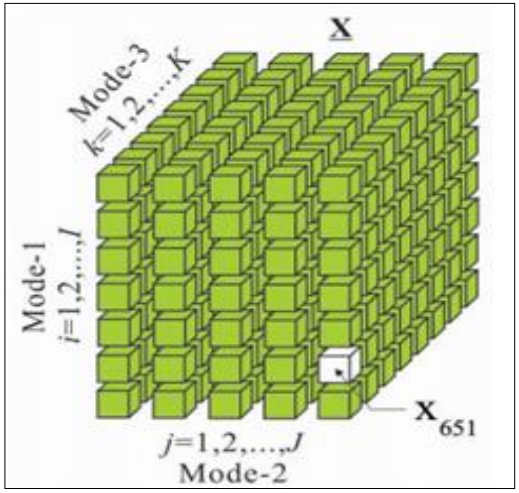
1.1. Examples: Image sequence represented as third-order tensor

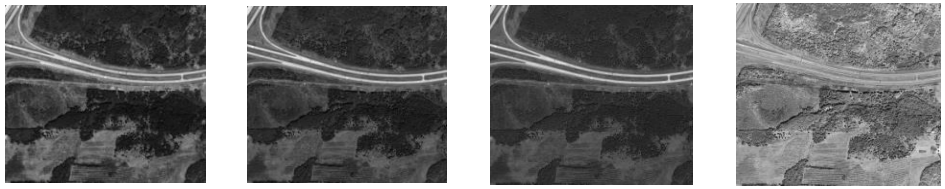
Multi-View Images (MVI): tensor of size 256×256×8



Computer Tomography images (CTI): tensor of size 512×512×8

3D tensor of size I×J×K





Red Green Blue Pan

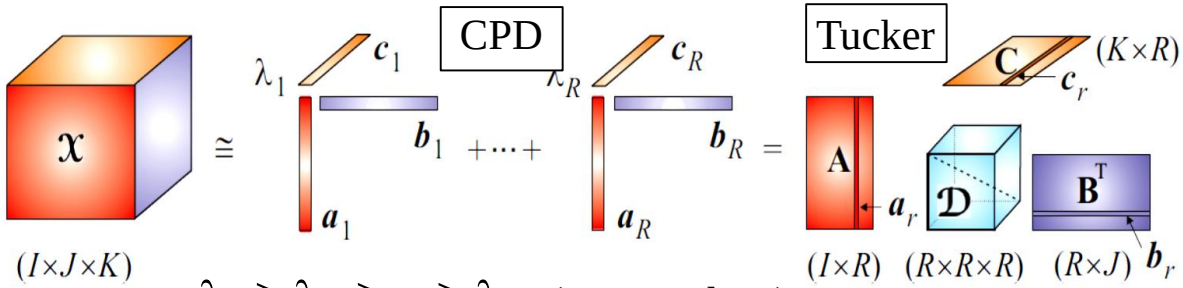
Multi-Spectral Images (MSI): tensor of size 512×512×4

Video sequence: tensor of size 256×256×8

1.2. Basic methods for tensor decomposition

1.2.1. Statistic methods: A. *Canonical Polyadic Decomposition* (CPD). (A. Cichocki et al. 2015)

3D tensor $\boldsymbol{X}$ approximated through a sum of R components (third-order tensors)

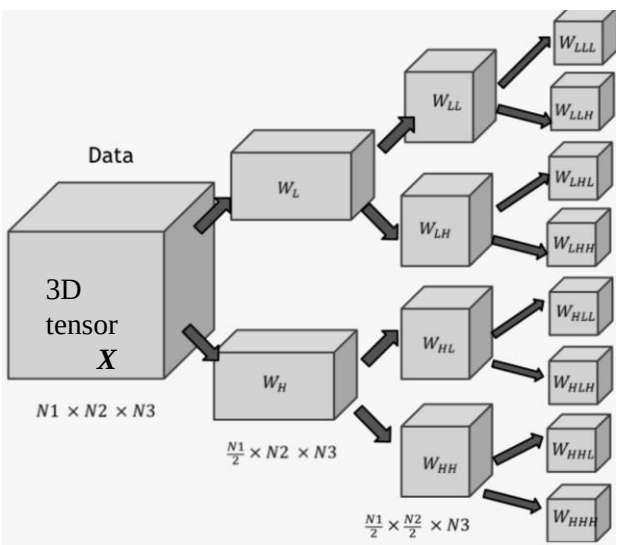


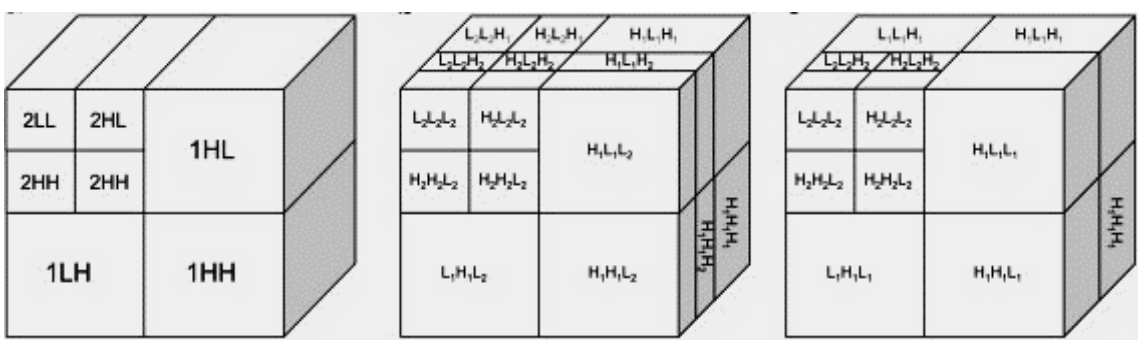
$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_R$ (eigen values)

Decomposition by CPD: $\boldsymbol{X} \approx \lambda_1(\vec{a}_1 \circ \vec{b}_1 \circ \vec{c}_1) + .. + \lambda_R(\vec{a}_R \circ \vec{b}_R \circ \vec{c}_R) = \sum_{r=1}^{R} \lambda_r(\vec{a}_r \circ \vec{b}_r \circ \vec{c}_r)$

B. *Tucker Decomposition*: $\boldsymbol{X} \approx \sum_{r_1=1}^{R_1}\sum_{r_2=1}^{R_2}\sum_{r_3=1}^{R_3} d_{r_1,r_2,r_3}(\vec{a}_{r_1} \circ \vec{b}_{r_2} \circ \vec{c}_{r_3})$ (outer product of 3 eigen vectors)

1.2.2. Deterministic methods: 3D Discrete Wavelet Transform (3D-DWT) (M. Vetterli et al. 2007)



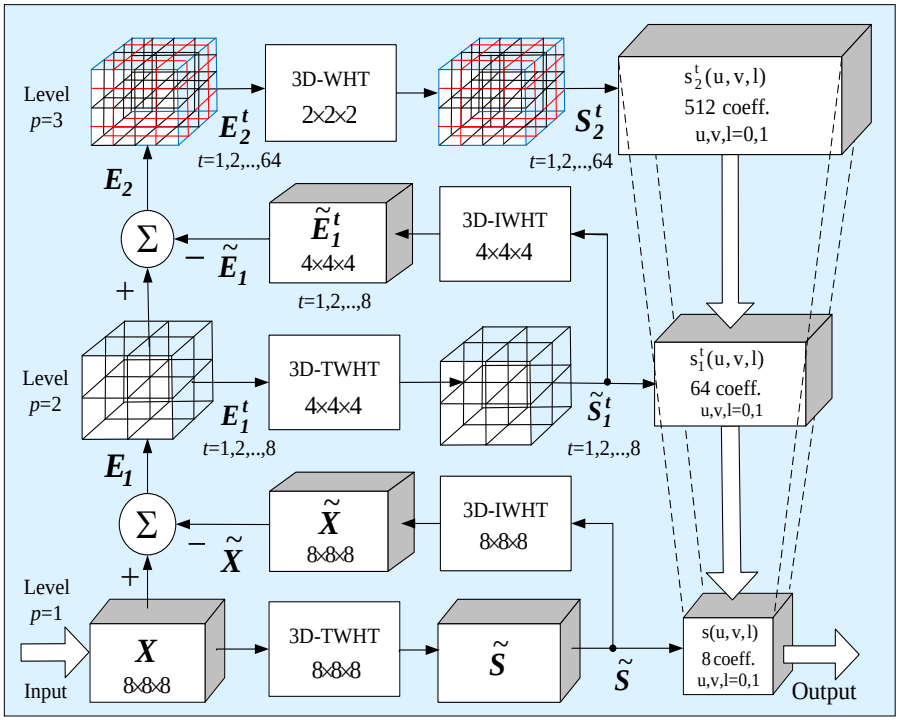


1D-DWT (direction z) 1D-DWT (direction y) 1D-DWT (direction x)

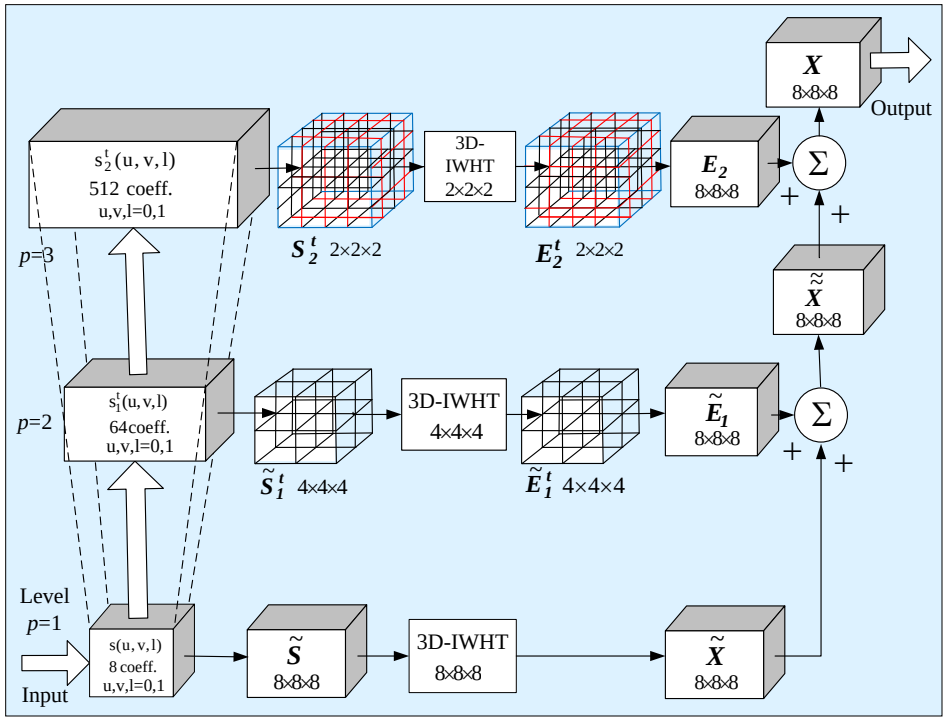
1.3. Proposed approach: decomposition of tensor $\boldsymbol{X}$ of size $N\times N\times N$ ($N=2^m$) by Inverse Spectrum Pyramid (ISP) based on 3D Walsh-Hadamard Transform (3D-WHT). Example: 3-level ISP for $N=2^3$.

3D-WHT transformation of the elements $x(i,j,k)$ of the tensor $\boldsymbol{X}$ into the coefficients $s(u,v,l)$ placed in the consecutive ISP levels.

Restoration of tensor $\boldsymbol{X}$ with elements $x(i,j,k)$ through inverse 3D-WHT, applied on the coefficients $s(u,v,l)$ in the ISP levels.



Structure of ISP for the decomposition of tensor $\boldsymbol{X}$ of size 8×8×8, based on the truncated 3D-WHT



Structure of inverse ISP for the restoration of tensor $\boldsymbol{X}$ of size 8×8×8, based on the truncated 3D-WHT

Disadvantage - the ISP is of the kind "overcomplete".

1.3.1. Definition of the 3D Walsh-Hadamard Transform (3D-WHT) for tensor of size $N \times N \times N$ ($N=2^m$) with elements $x(i,j,k)$: (S. Agaian et al., 2011)

Direct 3D-WHT:

$$s(u,v,l) = \sum_{i=0}^{N-1}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} x(i,j,k)\,wal(i,u,N)\,wal(j,v,N)\,wal(k,l,N)$$

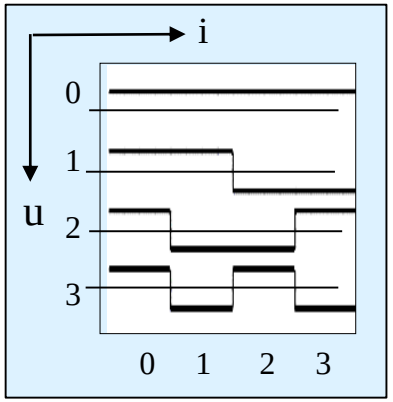
Inverse 3D-WHT:

$$x(i,j,k) = \frac{1}{N^3}\sum_{u=0}^{N-1}\sum_{v=0}^{N-1}\sum_{l=0}^{N-1} s(u,v,l)\,wal(i,u,N)\,wal(j,v,N)\,wal(k,l,N)$$

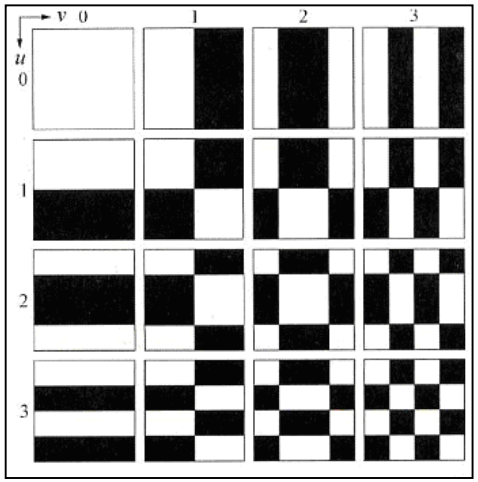
where $s(u,v,l)$ are 3D-WHT coefficients

1.3.2. Examples: 1D, 2D, and 3D WHT functions for N=4

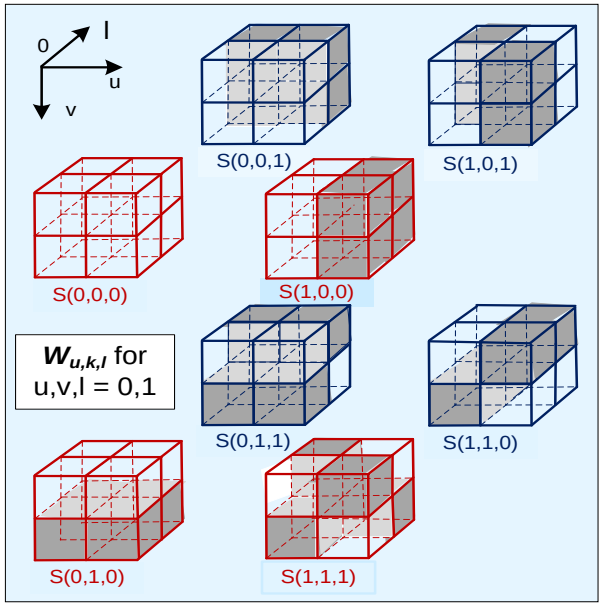
1D-WHT functions
$wal(i,u,4)$



2D-WHT functions
$wal(i,u,4)wal(j,v,4)$

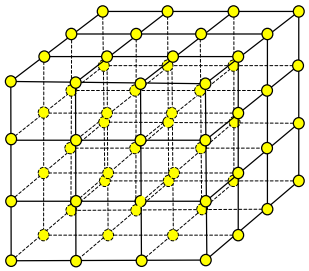


3D-WHT functions
$wal(i,u,4)wal(j,v,4)wal(k,l,4)$



2. Method for 3D-tensor representation through Reduced ISP

2.1. Decomposition of tensor *X* of size 4×4×4 through 3D Reduced ISP (3D-RISP):



$$[X_1]=\left[\begin{array}{cc|cc} b_{1,1} & b_{1,2} & b_{1,5} & b_{1,6} \\ b_{1,3} & b_{1,4} & b_{1,7} & b_{1,8} \\ \hline b_{1,9} & b_{1,10} & b_{1,13} & b_{1,14} \\ b_{1,11} & b_{1,12} & b_{1,15} & b_{1,16} \end{array}\right], \quad V_1 \leftarrow [X_2]=\left[\begin{array}{cc|cc} b_{2,1} & b_{2,2} & b_{2,5} & b_{2,6} \\ b_{2,3} & b_{2,4} & b_{2,7} & b_{2,8} \\ \hline b_{2,9} & b_{2,10} & b_{2,13} & b_{2,14} \\ b_{2,11} & b_{2,12} & b_{2,15} & b_{2,16} \end{array}\right],$$

$$[X_3]=\left[\begin{array}{cc|cc} b_{3,1} & b_{3,2} & b_{3,5} & b_{3,6} \\ b_{3,3} & b_{3,4} & b_{3,7} & b_{3,8} \\ \hline b_{3,9} & b_{3,10} & b_{3,13} & b_{3,14} \\ b_{3,11} & b_{3,12} & b_{3,15} & b_{3,16} \end{array}\right], \quad V_5 \leftarrow [X_4]=\left[\begin{array}{cc|cc} b_{4,1} & b_{4,2} & b_{4,5} & b_{4,6} \\ b_{4,3} & b_{4,4} & b_{4,7} & b_{4,8} \\ \hline b_{4,9} & b_{4,10} & b_{4,13} & b_{4,14} \\ b_{4,11} & b_{4,12} & b_{4,15} & b_{4,16} \end{array}\right],$$

For the level *p*=0 of the 3D-RISP are calculated the following 8 coefficients:

$$\alpha=s(0,0,0)=V_1+V_2+V_3+V_4+V_5+V_6+V_7+V_8; \qquad \eta=s(0,0,1)=V_1+V_2+V_3+V_4-V_5-V_6-V_7-V_8;$$

$$\beta=s(1,0,0)=V_1-V_2+V_3-V_4+V_5-V_6+V_7-V_8; \qquad \chi=s(1,0,1)=V_1-V_2+V_3-V_4-V_5+V_6-V_7+V_8;$$

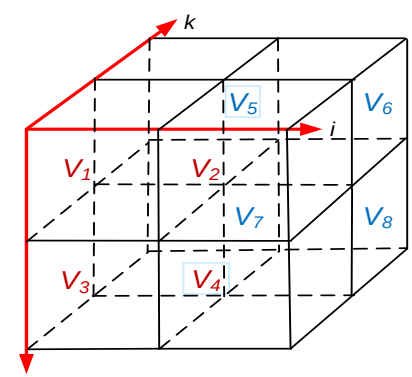
$$\gamma=s(0,1,0)=V_1+V_2-V_3-V_4+V_5+V_6-V_7-V_8; \qquad \rho=s(0,1,1)=V_1+V_2-V_3-V_4-V_5-V_6+V_7+V_8;$$

$$\delta=s(1,1,1)=V_1-V_2-V_3+V_4-V_5+V_6+V_7-V_8; \qquad \xi=s(1,1,0)=V_1-V_2-V_3+V_4+V_5-V_6-V_7+V_8.$$

where

$$V_1=\sum_{k=1}^{2}\sum_{i=1}^{4} b_{k,i}; \quad V_2=\sum_{k=1}^{2}\sum_{i=5}^{8} b_{k,i}; \quad V_3=\sum_{k=1}^{2}\sum_{i=9}^{12} b_{k,i}; \quad V_4=\sum_{k=1}^{2}\sum_{i=13}^{16} b_{k,i};$$

$$V_5=\sum_{k=3}^{4}\sum_{i=1}^{4} b_{k,i}; \quad V_6=\sum_{k=3}^{4}\sum_{i=5}^{8} b_{k,i}; \quad V_7=\sum_{k=3}^{4}\sum_{i=9}^{12} b_{k,i}; \quad V_8=\sum_{k=3}^{4}\sum_{i=13}^{16} b_{k,i}.$$



The first approximation $\tilde{\boldsymbol{X}}$ of the restored tensor $\boldsymbol{X}$:

$$\tilde{\boldsymbol{X}}=(1/4^3)[s(0,0,0)\boldsymbol{W}_{0,0,0}+s(1,0,0)\boldsymbol{W}_{1,0,0}+s(0,1,0)\boldsymbol{W}_{0,1,0}+s(1,1,1)\boldsymbol{W}_{1,1,1}+$$
$$+s(0,0,1)\boldsymbol{W}_{0,0,1}+s(1,0,1)\boldsymbol{W}_{1,0,1}+s(0,1,1)\boldsymbol{W}_{0,1,1}+s(1,1,0)\boldsymbol{W}_{1,1,0}].$$

where $\boldsymbol{W}_{u,v,l}$ are the basic 3D-WHT functions (tensors of size 4×4×4).

$$[\tilde{X}_\kappa]=\left[\begin{array}{cc:cc} A & A & B & B \\ A & A & B & B \\ \hdashline C & C & D & D \\ C & C & D & D \end{array}\right] \text{ for } k=1,2; \qquad [\tilde{X}_k]=\left[\begin{array}{cc:cc} F & F & G & G \\ F & F & G & G \\ \hdashline P & P & R & R \\ P & P & R & R \end{array}\right] \text{ for } k=3,4;$$

$A=(1/64)(\alpha+\beta+\gamma+\delta+\eta+\chi+\rho+\xi)=(1/8)V_1$; $\quad F=(1/64)(\alpha+\beta+\gamma-\delta-\eta-\chi-\rho+\xi)=(1/8)V_5$;

$B=(1/64)(\alpha-\beta+\gamma-\delta+\eta-\chi+\rho-\xi)=(1/8)V_2$; $\quad G=(1/64)(\alpha-\beta+\gamma+\delta-\eta+\chi-\rho-\xi)=(1/8)V_6$;

$C=(1/64)(\alpha+\beta-\gamma-\delta+\eta+\chi-\rho-\xi)=(1/8)V_3$; $\quad P=(1/64)(\alpha+\beta-\gamma+\delta-\eta_1-\chi+\rho-\xi)=(1/8)V_7$;

$D=(1/64)(\alpha-\beta-\gamma+\delta+\eta-\chi-\rho+\xi)=(1/8)V_4$; $\quad R=(1/64)(\alpha-\beta-\gamma-\delta-\eta+\chi+\rho+\xi)=(1/8)V_8$.

Calculation of the difference tensor $\boldsymbol{E}_0=\boldsymbol{X}-\tilde{\boldsymbol{X}}$, represented by the 4 matrices below:

$$[E_{0,1}]=\left[\begin{array}{cc:cc} b_{1,1}-A & b_{1,2}-A & b_{1,5}-B & b_{1,6}-B \\ b_{1,3}-A & b_{1,4}-A & b_{1,7}-B & b_{1,8}-B \\ \hdashline b_{1,9}-C & b_{1,10}-C & b_{1,13}-D & b_{1,14}-D \\ b_{1,11}-C & b_{1,12}-C & b_{1,15}-D & b_{1,16}-D \end{array}\right], \qquad [E_{0,2}]=\left[\begin{array}{cc:cc} b_{2,1}-A & b_{2,2}-A & b_{2,5}-B & b_{2,6}-B \\ b_{2,3}-A & b_{2,4}-A & b_{2,7}-B & b_{2,8}-B \\ \hdashline b_{2,9}-C & b_{2,10}-C & b_{2,13}-D & b_{2,14}-D \\ b_{2,11}-C & b_{2,12}-C & b_{2,15}-D & b_{2,16}-D \end{array}\right],$$

$$[E_{0,3}]=\left[\begin{array}{cc:cc} b_{3,1}-F & b_{3,2}-F & b_{3,5}-G & b_{3,6}-G \\ b_{3,3}-F & b_{3,4}-F & b_{3,7}-G & b_{3,8}-G \\ \hdashline b_{3,9}-P & b_{3,10}-P & b_{3,13}-R & b_{3,14}-R \\ b_{3,11}-P & b_{3,12}-P & b_{3,15}-R & b_{3,16}-R \end{array}\right], \qquad [E_{0,4}]=\left[\begin{array}{cc:cc} b_{4,1}-F & b_{4,2}-F & b_{4,5}-G & b_{4,6}-G \\ b_{4,3}-F & b_{4,4}-F & b_{4,7}-G_1 & b_{4,8}-G \\ \hdashline b_{4,9}-P & b_{4,10}-P & b_{4,13}-R & b_{4,14}-R \\ b_{4,11}-P & b_{4,12}-P & b_{4,15}-R & b_{4,16}-R \end{array}\right],$$

For the level p=1 of the 3D-RISP the tensor $\boldsymbol{E}_0$ is divided into 8 sub-tensors $\boldsymbol{E}_0^t$ of size $2\times2\times2$ where t=1,2,..,8. For each sub-tensor are calculated $V_{t1}\div V_{t8}$.

For the sub-tensor $\boldsymbol{E}_0^1$ (when t=1) are calculated $V_{11}\div V_{18}$:

$V_{11}=b_{1,1}-A$; $V_{12}=b_{1,2}-A$; $V_{13}=b_{1,3}-A$; $V_{14}=b_{1,4}-A$;

$V_{15}=b_{2,1}-A$; $V_{16}=b_{2,2}-A$; $V_{17}=b_{2,3}-A$; $V_{18}=b_{2,4}-A$.

$$\boxed{\alpha_1^1=s_1(0,0,0)=V_{11}+V_{12}+V_{13}+V_{14}+V_{25}+V_{16}+V_{17}+V_{18}=V_1-8A=0;}$$

$$\beta_1^1=s_1(1,0,0)=V_{11}-V_{12}+V_{13}-V_{14}+V_{15}-V_{16}+V_{17}-V_{18};$$

$$\gamma_1^1=s_1(0,1,0)=V_{11}+V_{12}-V_{13}-V_{14}+V_{15}+V_{16}-V_{17}-V_{18};$$

$$\delta_1^1=s_1(1,1,1)=V_{11}-V_{12}-V_{13}+V_{14}-V_{15}+V_{16}+V_{17}-V_{18};$$

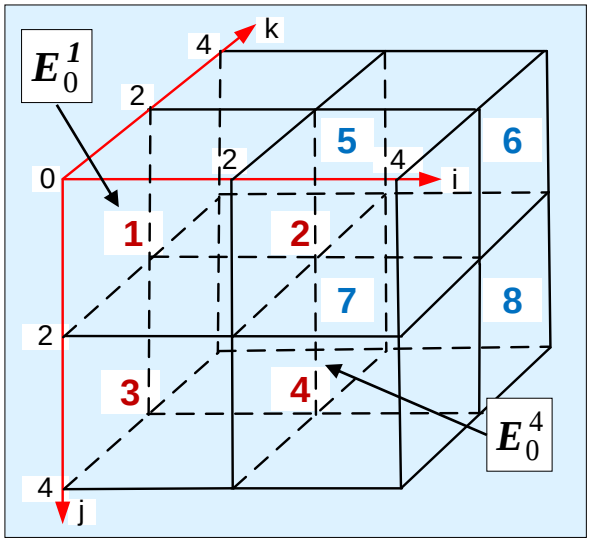
$$\eta_1^1=s_1(0,0,1)=V_{11}+V_{12}+V_{13}+V_{14}-V_{15}-V_{16}-V_{17}-V_{18};$$

$$\chi_1^1=s_1(1,0,1)=V_{11}-V_{12}+V_{13}-V_{14}-V_{15}+V_{16}-V_{17}+V_{18};$$

$$\rho_1^1=s_1(0,1,1)=V_{11}+V_{12}-V_{13}-V_{14}-V_{15}-V_{16}+V_{17}+V_{18};$$

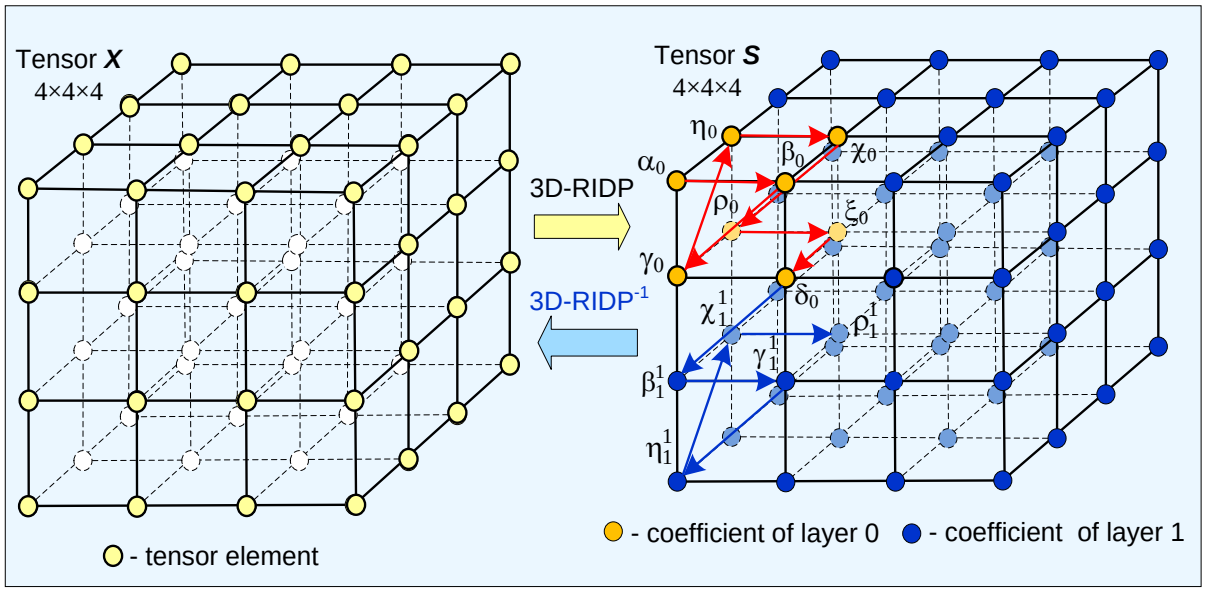
$$\xi_1^1=s_1(1,1,0)=V_{11}-V_{12}-V_{13}+V_{14}+V_{15}-V_{16}-V_{17}+V_{18}.$$

Division of tensor $\boldsymbol{E}_0$ into 8 sub-tensors $\boldsymbol{E}_0^t$, for t=1,2,..,8



In similar way are calculated the remaining cofficients $s_t(u,v,l)$, for the sub-tensors $\boldsymbol{E}_0^t$ of size 2×2×2, for t = 2,3,...,8. Then, $\boxed{\alpha_1^t = s_t(0,0,0) = V_t - 8A_t = 0.}$

In this case, the number of the coefficients for the level 1 is 8, and for the level 2 it is 56, hence - the 3D-RISP is "non-overcomplete" decomposition - it has 64 coefficients only (4×4×4).

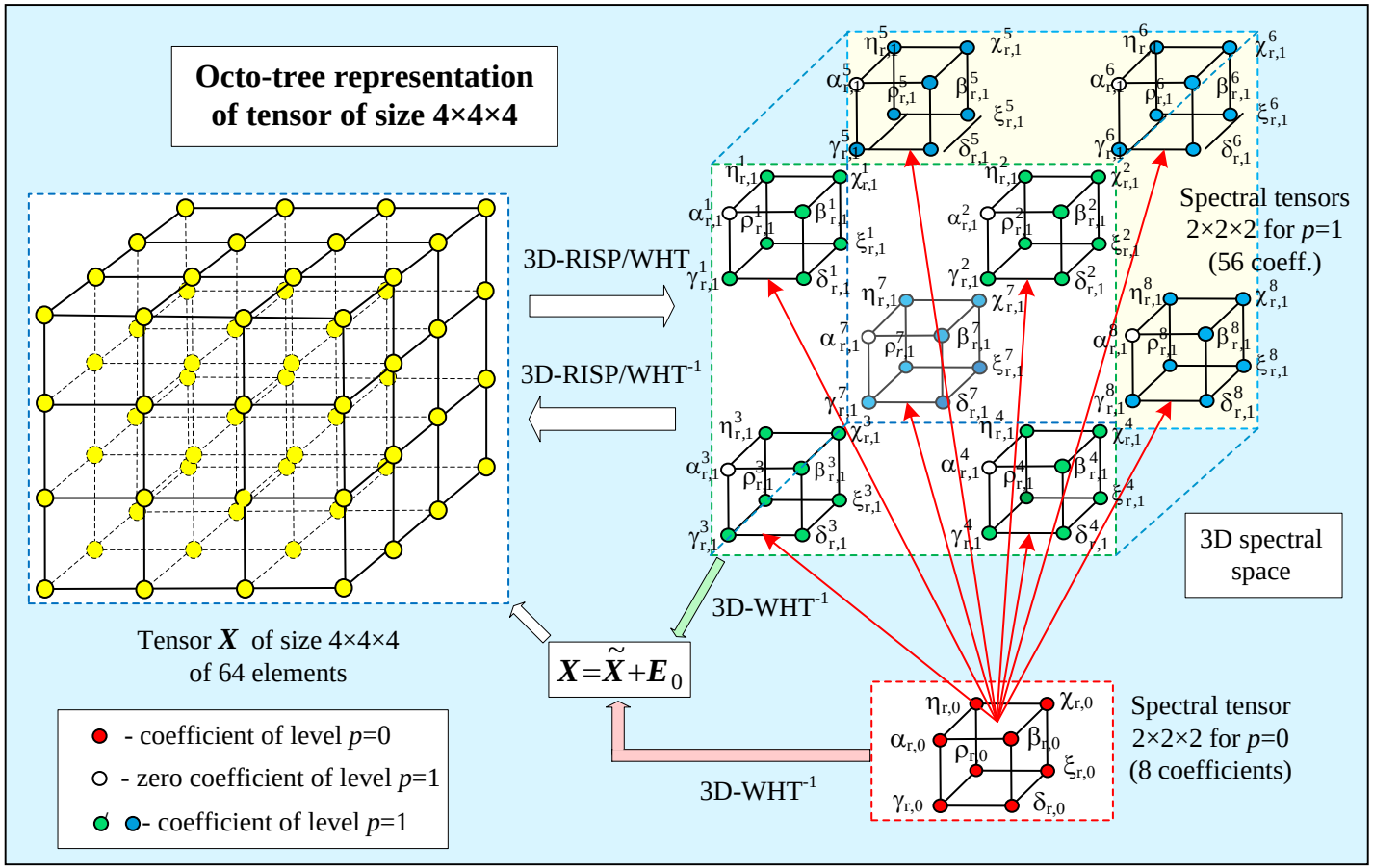


Transformation of the tensor ***X*** of size 4×4×4 into the spectrum tensor ***S*** of same size by using the 3D-RISP of 2 levels (8+56 coefficients)

The elements 1,2,.,64 of the tensor ***S*** are transformed into vectors, as follows:

$$\vec{S} = [s_1, s_2, .., s_{64}]^T \quad \text{where} \quad |\bar{s}_1| >> |\bar{s}_2| >> .. >> |\bar{s}_{64}|$$

Applications: Image tensor compression, CBIR in a tensor DB, etc.



2.2. Generalized decomposition: 3D-RISP/WHT for a tensor X of size $2^m \times 2^m \times 2^m$

$$\boxed{\boldsymbol{X}=\tilde{\boldsymbol{X}}+\sum_{p=0}^{m-3}\tilde{\boldsymbol{E}}_p+\boldsymbol{E}_{m-2},}$$

where $\tilde{\boldsymbol{X}}=(1/2^{3m})\sum_{u=0}^{1}\sum_{v=0}^{1}\sum_{l=0}^{1}s(u,v,l)\boldsymbol{W}_{u,v,l}$ for $p=0$; $\tilde{\boldsymbol{E}}_p=\bigcup_{t=1}^{2^{mp}}\tilde{\boldsymbol{E}}_p^t$

$$\tilde{\boldsymbol{E}}_p^t=1/2^{3(k-m)}\sum_{u=0}^{1}\sum_{v=0}^{1}\sum_{l=0}^{1}s_p^t(u,v,l)\boldsymbol{W}_{u,v,l} \text{ for } p=1,2,..,m\text{-}1 \text{ and } t=1,2,..,2^{mp};\ \boldsymbol{E}_0=\boldsymbol{X}-\tilde{\boldsymbol{X}};\ \boldsymbol{E}_p^t=\boldsymbol{E}_{p-1}^t-\tilde{\boldsymbol{E}}_{p-1}^t$$

3. Main properties of the 3D spectrum tensor obtained through RISP/WHT

3.1. Relations between powers P_1 and P_2 of the coefficients in the first and second pyramid levels, when m=2:

$$\psi=\frac{P_1}{P_2}=\frac{8(\alpha^2+\beta^2+\gamma^2+\delta^2+\eta^2+\chi^2+\rho^2+\xi^2)}{\sum_{t=1}^{8}(\beta_{t1}^2+\gamma_{t1}^2+\delta_{t1}^2+\eta_{t1}^2+\chi_{t1}^2+\rho_{t1}^2+\xi_{t1}^2)}>>1,\ \text{because}\ \ \alpha^2=(\sum_{r=1}^{8}V_r)^2>>(\beta^2+\gamma^2+\delta^2+\eta^2+\chi^2+\rho^2+\xi^2)$$

and $8(\alpha^2+\beta^2+\gamma^2+\delta^2+\eta^2+\chi^2+\rho^2+\xi^2)>>\sum_{t=1}^{8}(\beta_{t1}^2+\gamma_{t1}^2+\delta_{t1}^2+\eta_{t1}^2+\chi_{t1}^2+\rho_{t1}^2+\xi_{t1}^2)$

3.2. Number of the coefficients of the spectrum tensor S of size $N\times N\times N$ for $N=2^m$ in the m-level decomposition:

$N_\Sigma=8+7\times8+7\times8^2+..+7\times8^{m-1}=8+7\sum_{p=1}^{m-1}8^p=8^m=N^3$. Hence, RISP is "**non-overcomplete**".

3.3. Relative number of the retained coefficients in a pyramid of $p\le m$ levels in respect of N_Σ

$\tau(p)=8^p/N_\Sigma=1/8^{m-p}$ for p=1,2,..,m.

Level (p)	1	2	3	……	m-2	m-1	m
$\tau(p)$	$1/8^{m-1}$	$1/8^{m-2}$	$1/8^{m-3}$	.....	$1/8^2$	1/8	1

3.4. Mean square error $\varepsilon(p)$ for the approximated restored tensor, $\boldsymbol{X}$ is:

$$\varepsilon(p)=\begin{cases}(1/N^3)\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}e_p(i,j,k)^2=(1/N^3)\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}[x(i,j,k)-\tilde{x}_p(i,j,k)]^2 & \text{for } p=1,2,..,m-1,\\ 0 \quad - \quad \text{for} & p=m.\end{cases}$$

4. Computational complexity for a 3D-RISP/WHT

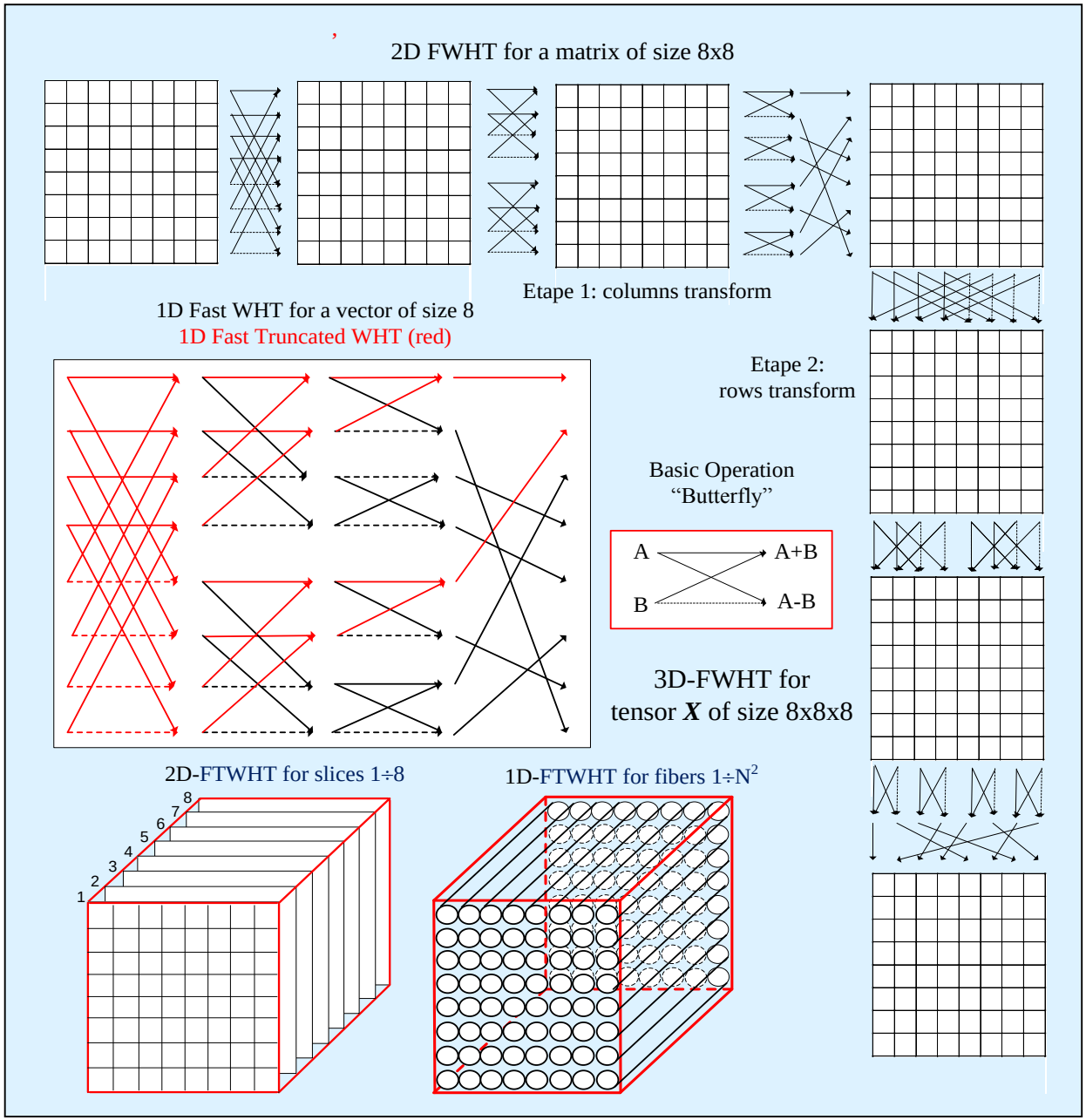
4.1. Computational complexity (CC) of 3D-RISP/WHT with *m*=2 levels, based on number of additions (A) and multiplications (M):

3D-RISP parameters for m=2	A	M
For level p=0		
$V_1 \div V_8$	8×7=56	0
$\alpha \div \xi$	8×7=56	0
$A \div R$	0	8
$\tilde{\boldsymbol{X}}$	0	1
$[E_{0,k}]$ for k=1,2,3,4	16×4=64	0
For level p=1		
$V_{t1} \div V_{t8}$, t=1,2,..,8	8×8=64	0
$\beta_{t1} \div \xi_{t1}$, t=1,2,..,8	8×7×7=392	0
Total number	632	9

The multiplication by 1/8 is not the "classic" operation (it is just a binary number translation). Hence, practically the needed operations are only additions (632), without any multiplications.

4.2. Computational complexity (CC) of pyramid of *m* levels, based on the "fast" 3D-WHT:

4.2.1.Algorithm for Fast Truncated 3D-WHT (3D-FTWHT)



The 3D-WHT is a separable. Number of "additions" A_F for the "Fast" 1D-WHT (1D-FWHT):

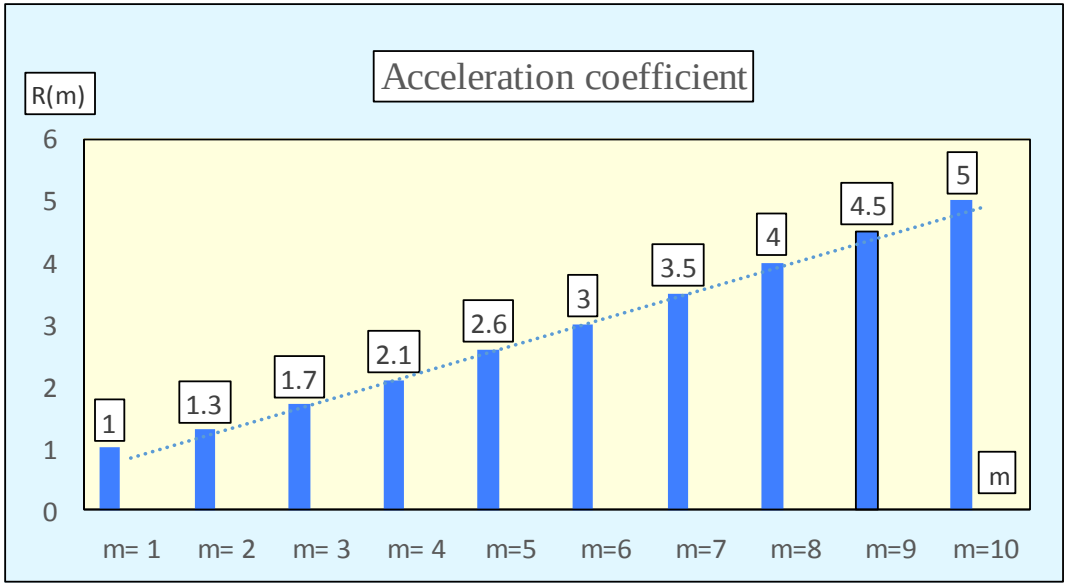
(Gonzales and Woods, 2008)

$$A_F(m)=N\lg_2 N=m2^m.$$

Number of the operations "addition" A_{FT} for the "Fast Truncated" 1D-WHT (1D-FTWHT), when a reduction (truncation) of the output coefficients from N down to 2:

$$A_{FT}(m)=N\sum_{p=0}^{m-1}2^{-p}=2(2^m-1).$$

4.2.2. Acceleration of the calculations: $R_{1D}(m)=\dfrac{A_F(m)}{A_{FT}(m)}=\dfrac{m}{2(1-2^{-m})}$ ⇒ $R_{1D}(m)\approx m/2$



Total number of A_{FT} for tensor of size $N\times N\times N$, transformed by a 3D-FTWHT: $A_{FT}^{3D}(N)=2N^2(N\lg_2 N+N-1)$

Total number of A for tensor of size $N\times N\times N$, transformed by a 3D-FWHT: $A^{3D}(N)=3N^3(N-1)$

4.2.3. Acceleration of the calculations for 3D-FTWHT compared to 3D-FWHT and 3D-WHT:

$$R_{3D}(N)=\frac{A_F}{A_{FT}}=\frac{3Nm}{2[N(m+1)-1]}\approx\frac{1.5m}{m+1}.$$ For $m=4$ $R_{3D}(16)=1.2$;

$$R'_{3D}(N)=\frac{A}{A_{FT}}=\frac{3N(N-1)}{2[N(m+1)-1]}\approx\frac{1.5\times 2^m}{m+1}.$$ For $m=4$ $R'_{3D}(16)=4.8$.

4.2.4. Computational complexity of 3D-RISP/WHT

For p=0 the number of additions/multiplications: $A_0 = 2^{2m+1}[2^m(m+1)-1]+3m2^{3m}$; $M_0=1$

- -

For $p=p$ the number of additions/multiplications: $A_p=2^{2m+1+p}[2^{m-p}(m-p+1)-1]+3(m-p)2^{3m}$; $M_p=2^{3p}$;

- -

For $p=m$-2 the number of additions/multiplications: $A_{m-2}=11\times 2^{3m}$ $M_{m-2}=2^{3(m-2)}$

For $p=m$-1 the number of additions/multiplications: $A_{m-1}=3\times 2^{3m}$ $M_{m-1}=0$

Total number of additions/multiplications:

$$A_{3D-FRISP}(m)=\sum_{p=0}^{m-1} A_p=2^{3m}\{(m-1)[(5/2)m+7)]+4\}; \quad M_{3D-RISP}(m)=\sum_{p=0}^{m-1} M_p=(1/7)[2^{3(m-1)}-1].$$

Example: for m=2 $A_{3D-FRISP}(2)=\sum_{p=0}^{1} A_p=768$; $A_{3D-RISP}(2)=\sum_{p=0}^{1} A_p=5376$.

Acceleration на изчисляването на 3D-FRISP спрямо 3D-RISP по отношение на A(m) и M(m) е:

For tensor with N^3 voxels: $A_{3D\text{-}FRISP}\approx 2.5m^2N^3$ $M_{3D\text{-}FRISP}\approx(1/56)N^3$.

4.2.5. Computational complexity of 3D-DWT

The 3D Wavelet pyramid (WP) with m levels for tensor of size $N \times N \times N$ ($N=2^m$) using a bank of separable digital filters with 3 and 5 coefficients (1/2)(1,2,1) and (1/8)(-1,2,6,2,-1) requires:

For level p=0 the number of additions and multiplications: $A_0=18N^3$; $M_0=24N^3$;

For level p=1 the number of additions and multiplications: $A_1=18N^3(1/2^3)$; $M_1=24N^3(1/2^3)$;

- -

For level $p=p$ the number of additions and multiplications: $A_p=18N^3(1/2^{3p})$; $M_p=24N^3(1/2^{3p})$;

- -

For level $p=m$-1 the number of additions and multiplications: $A_{m-1}=18N^3(1/2^{3m-3})$; $M_{m-1}=24N^3(1/2^{3m-1})$;

Total number of additions and multiplications:

$$A_{3D-DWT}(m)=\sum_{p=0}^{m-1} A_p=18N^3\sum_{p=0}^{m-1}8^{-p}=20.5(N^3-1); \quad M_{3D-DWT}(m)=\sum_{p=0}^{m-1} M_p=24N^3\sum_{p=0}^{m-1}8^{-p}=27.4(N^3-1)$$

4.3. Comparison of the computational complexity of decompositions

3D Tensor Decomposition	$\sum_{p=0}^{m-1} A_p$	$\sum_{p=0}^{m-1} M_p$	$R=O_{3D-DWT}/O_{3D-RISP}$
3D-RISP/WHT	$2.5m^2N^3$	$(1/56)N^3$	$R_A(m)=8.23/m^2$
3D-DWT	$20.5N^3$	$27.4N^3$	$R_M=1534$

5. Numerical example: tensor of size 8×8×8 transformed by 3D-RISP/WHT

The average change between the elements of the successive matrices $[X_k]$ is 20 %.

$$[X_1]=\left[\begin{array}{cccc:cccc}
1&1&3&3&2&3&2&2\\
2&2&6&7&7&7&3&2\\
1&1&6&7&7&7&3&2\\
2&3&3&7&8&8&2&2\\
\hdashline
1&2&2&3&4&4&4&3\\
3&2&2&2&1&1&3&2\\
1&1&2&2&2&1&1&1\\
1&1&3&3&3&2&2&2
\end{array}\right];
[X_2]=\left[\begin{array}{cccc:cccc}
1&1&3&3&2&3&2&2\\
2&2&7&7&7&6&3&2\\
1&1&7&7&7&6&3&2\\
2&3&7&8&8&7&2&2\\
\hdashline
1&2&2&3&4&4&4&3\\
3&2&2&2&1&1&3&2\\
1&1&2&2&2&2&2&2\\
1&1&3&3&3&2&2&3
\end{array}\right];
[X_3]=\left[\begin{array}{cccc:cccc}
1&1&3&3&2&3&2&2\\
2&7&7&7&7&3&3&2\\
1&7&7&7&7&3&3&2\\
2&7&7&8&8&2&2&2\\
\hdashline
1&2&2&3&4&4&4&3\\
3&2&2&2&2&2&3&2\\
1&1&2&2&2&3&3&3\\
1&1&3&3&3&3&3&4
\end{array}\right];
[X_4]=\left[\begin{array}{cccc:cccc}
1&1&3&3&2&3&2&2\\
2&7&7&7&6&6&3&2\\
1&7&7&7&6&6&3&2\\
2&7&7&8&7&3&2&2\\
\hdashline
1&2&2&3&4&4&4&3\\
3&2&2&2&1&3&3&2\\
1&1&2&2&3&3&4&4\\
1&1&3&3&3&3&4&5
\end{array}\right];$$

$$[X_5]=\left[\begin{array}{cccc:cccc}
1&1&3&3&2&3&2&2\\
7&7&7&6&6&6&3&2\\
7&7&7&6&6&6&3&2\\
7&7&8&7&6&3&2&2\\
\hdashline
1&2&2&3&4&4&4&3\\
3&2&2&2&1&3&4&4\\
1&1&2&2&3&4&5&5\\
1&1&2&3&3&4&5&6
\end{array}\right];
[X_6]=\left[\begin{array}{cccc:cccc}
1&1&3&3&2&3&2&2\\
7&7&6&6&6&6&3&2\\
7&7&6&6&6&6&3&2\\
7&8&7&6&6&3&2&2\\
\hdashline
1&2&2&3&4&4&4&3\\
3&2&2&1&2&4&5&5\\
1&1&2&3&4&5&6&6\\
1&1&2&3&4&5&6&7
\end{array}\right];
[X_7]=\left[\begin{array}{cccc:cccc}
1&1&3&3&2&3&2&2\\
7&6&6&6&6&3&2&1\\
7&6&6&6&6&3&2&1\\
8&7&7&6&4&3&2&1\\
\hdashline
1&2&2&3&4&4&4&3\\
3&2&2&1&2&4&5&6\\
1&1&2&3&4&5&6&7\\
1&1&2&4&5&6&7&8
\end{array}\right];
[X_8]=\left[\begin{array}{cccc:cccc}
1&1&3&4&2&3&2&1\\
6&6&6&6&5&3&2&1\\
6&6&6&6&5&3&2&1\\
7&7&6&6&4&3&2&1\\
\hdashline
1&1&2&3&4&2&2&3\\
2&2&1&1&3&4&5&6\\
1&1&3&3&4&5&6&7\\
1&1&3&4&5&6&7&8
\end{array}\right].$$

Level *p*=0 of 3D-RISP/WHT (8 coefficients)

α	β	γ	δ	η	χ	ρ	ξ
1770	- 42	340	- 242	- 144	- 4	82	408

$$P_0=(1/8)\sum_{u=0}^{1}\sum_{v=0}^{1}\sum_{l=0}^{1}s(u,v,l)^2=(1/8)(\alpha^2+\beta^2+\gamma^2+\delta^2+\eta^2+\chi^2+\rho^2+\xi^2)=437846$$

$\alpha_p = 0$ for $p=1,2,..,8$

p	β_p	γ_p	δ_p	η_p	χ_p	ρ_p	ξ_p
1	- 107	-45	5	-57	-23	15	17
2	96	-30	-12	22	20	-10	-32
3	-29	13	1	-1	-1	1	21
4	-9	7	-13	-25	5	17	1
5	-4	-78	4	10	2	-4	-22
6	73	-11	-3	29	7	-5	-13
7	-29	3	-5	-1	5	9	25
8	-41	-55	-1	-21	3	15	1

Level 2 of the 3D-RISP/WHT (56 coefficients)

$$P_2 = (1/64)\sum_{p=1}^{8}(\beta_p^2+\gamma_p^2+\delta_p^2+\eta_p^2+\chi_p^2+\rho_p^2+\xi_p^2)=865$$

Level 3 of the 3D-RISP/WHT (448 coefficients)

$$P_3 = (1/512)\sum_{p=1}^{64}(\beta_p^2+\gamma_p^2+\delta_p^2+\eta_p^2+\chi_p^2+\rho_p^2+\xi_p^2)=11$$

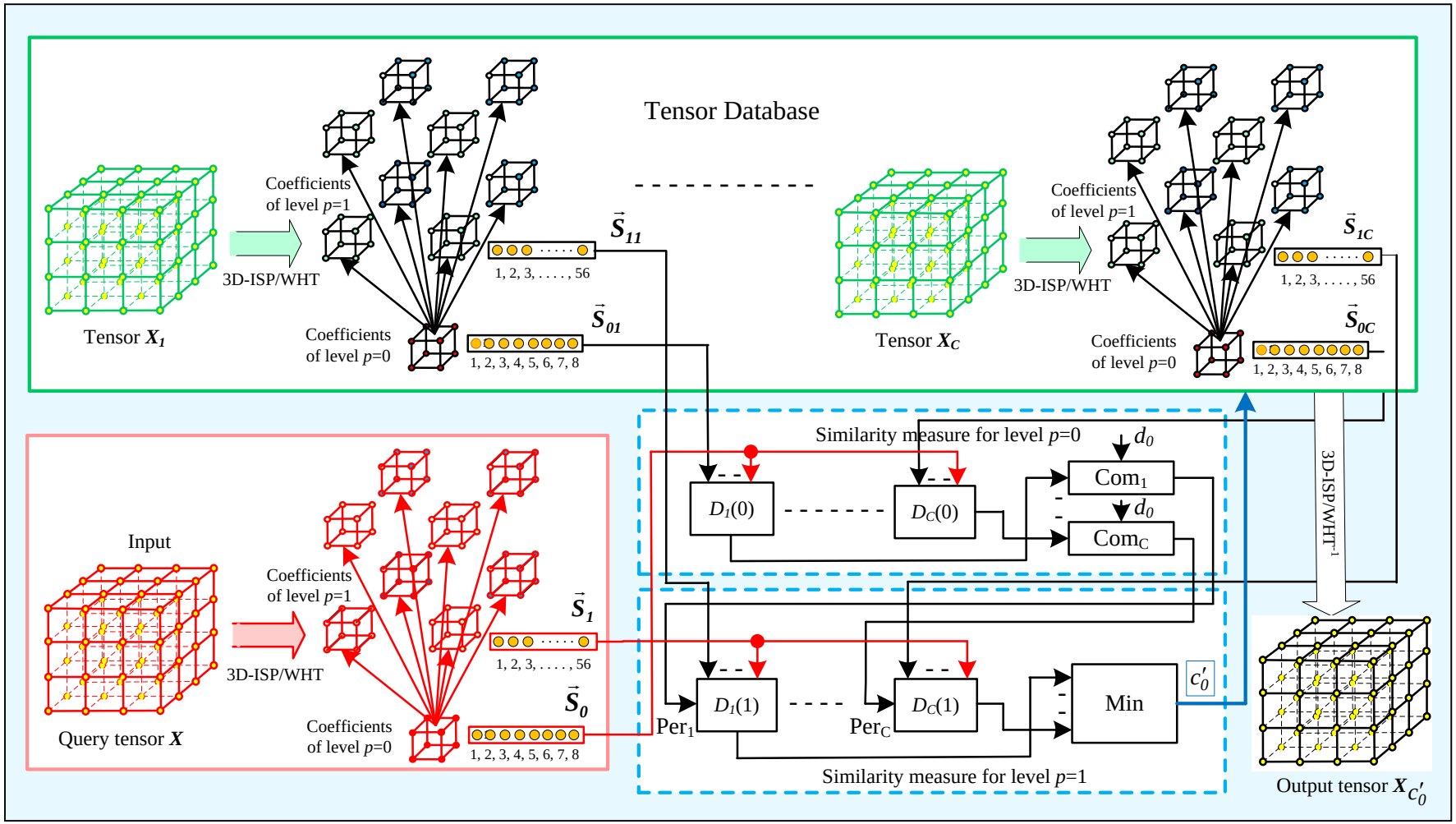
The total number of the spectrum coefficients equal to zero: 200 from 512, or 39%.

The relation between the normalized powers: $P_{10}:P_{20}:P_{30}=1 : 0.002 : 0.00003$.

In case that the coefficients in the third 3D-RISP level are ignored is got:

- Relation Signal/Noise: $SNR = 10\lg_{10}10[(P_1+P_2+P_3)/P_3] = 46$ dB (visually lossless);
- Compression coefficient: $K=512/72=7.53$.

6. Application of the 3D-RISP for content-based multi-level search in a tensor DB



Block diagram of the content-based multi-level search illustrated for two-level 3D-RISP/WHT

7. Conclusions

Main advantages of the tensor decomposition based on the 3D-RISP/WHT:

- *High concentration* of the tensor energy in a small number of spectrum coefficients placed in the initial level, which permits efficient information redundancy reduction;
- *Minimum computational complexity* regarding the famous tensor decompositions based on the multidimensional Wavelet Transform, PCA, SVD, and their variations. The main reasons are the divisibility of the 3D-WHT, the use of the "fast" 1D-WHT without „multiplication" (only "addition"), and the possibility for coefficients' s(0,0,0) number reduction in each group of 8 coefficients for one sub-tensor $\boldsymbol{E}_{\boldsymbol{p}}^{\boldsymbol{t}}$.

Future development of 3D-RISP/WHT structure:

- Investigation of various 3D decompositions, such as: the truncated pyramid, the locally-adaptive pyramid in respect of the sub-tensors contents, etc.;
- Extension for *N*-dimensional pyramid, applied on sequence of tensors (train tensor).

The future investigations on the 3D-RISP are aimed at the following areas:

- ✓ Development of algorithms for compression of image sequences, represented as 3D tensors by using other kinds of orthogonal transforms (DCT, Hartley, etc.);
- ✓ Application for content-based multi-level search in a tensor DB;
- ✓ Big data analysis, using 3D convolutional neural networks for deep learning, etc.

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Thank you for your attention